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VARIATIONAL AND EXTENDED SUMS OF MONOTONE OPERATORS[1]

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Abstract

In this article we show that the notion of variational sum of maximal monotone operators, introduced by Attouch, Baillon and Théra in [3] in the setting of Hilbert spaces, can be successfuly extended to the case of reflexive Banach spaces, preserving all of its properties. We make then a comparison with the usual pointwise sum and with the notion of extended sum proposed in our paper [26].

Keywords: monotone operators, sums of operators, enlargements, Yosida regularizations, subdifferentials

AMS subject classification: 47H05, 52A41, 26B25

1 Introduction

In [3], Attouch, Baillon and Théra introduced a notion of sum of two monotone operators, called *variational sum*. This was done in the setting of Hilbert spaces and the general idea was to use an appropriate approximation (the Yosida regularization) of a monotone operator in a Hilbert space and to take as a sum of two operators a suitable limit of the pointwise sum of their approximations. In several cases this sum appears to be larger than the usual pointwise sum, as for example in the case of subdifferentials of convex functions.

The need to look for a notion of sum of two monotone operators, which is a generalization of the usual pointwise sum, is inspired by the fact that sometimes (as above in the case of subdifferentials) the study of a problem, with monotone operators involved, leads to an operator that turns out to be larger than the pointwise sum (see e.g. [3, 4], where problems arising in partial differential equations possessing this property are studied). Therefore, different authors have tried to investigate possible generalized notions of sum of monotone operators

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like, the variational sum mentioned above, or the parallel sum used in [16] to study electrical networks, or a sum based on the Trotter-Lie formula in [17], or recently an accretive extension of the algebraic sum of two linear operators in [10].

Our aim in this article is twofold. First, to show that the notion of variational sum can be naturally extended to the more general setting of reflexive Banach spaces using the same general idea of approximation. And second, to compare this sum with a notion of extended sum introduced in our paper [26], based on another approximation of a given monotone operator–a suitable enlargement of the operator. Therefore, in the rest of the paper we proceed as follows: after some necessary preliminaries given in Section 2, in Section 3 we present the idea of the extended sum from [26]. Further, in Section 4 we show how the concept of variational sum can be extended to the case of a reflexive Banach space. We see that the generalized notion keeps its properties from Hilbert space setting. We prove also that, when the variational sum of two maximal monotone operators is maximal monotone, it contains the extended (and hence the pointwise) sum of the operators and all the three concepts coincide if we suppose the maximality of the pointwise sum. In the last Section 5, we see that another important property from the Hilbert space setting–the fact that the subdifferential of the sum of two convex proper lower semicontinuous functions is equal to the variational sum of their subdifferentials–is also preserved in the new setting. In particular, we obtain that the two concepts–the variational sum and the extended sum–agree also for the case of subdifferentials of convex functions.

2 Some preliminaries

Throughout this article we will assume that X is a reflexive real Banach space with continuous dual X^*. The norms in X and X^* will be designated by $\|\cdot\|$, w will mean the weak topology in X and X^*. For the usual pairing between X and X^* we use the symbol $\langle \cdot, \cdot \rangle$.

Due to a result of John and Zizler [14], we may think that X is endowed with a Fréchet differentiable (away from the origin) locally uniformly rotund norm whose dual is also Fréchet differentiable except the origin and locally uniformly rotund. In particular (see e.g. Diestel [11]), these norms are not only strictly convex, but also satisfy the Kadec-Klee property:

$$\text{if } x_n \to x \text{ weakly in } X \text{ and } \|x_n\| \to \|x\|, \text{ then } x_n \to x \text{ strongly in } X,$$

and

$$\text{if } x_n^* \to x^* \text{ weakly in } X^* \text{ and } \|x_n^*\| \to \|x^*\|, \text{ then } x_n^* \to x^* \text{ strongly in } X^*.$$

Evereywhere in the sequel, we will assume that the norms in X and X^* are as above.

Given a (multivalued) operator $A : X \rightrightarrows X^*$, as usual the *graph* of A will be denoted by

$$\mathrm{Gr}(A) := \{(x, x^*) \in X \times X^* : x^* \in Ax\}$$

and the *domain* and the *range* of A by

$$\mathrm{Dom}(A) := \{x \in X : \; Ax \neq \emptyset\}$$

and

$$\mathrm{R}(A) := \bigcup \{Ax : x \in \mathrm{Dom}(A)\}.$$

The inverse operator of $A : X \rightrightarrows X^*$ is $A^{-1} : X^* \rightrightarrows X$:

$$A^{-1}x^* := \{x \in X : x^* \in Ax\}, \; x^* \in X^*.$$

Obviously $\mathrm{Dom}(A^{-1}) = \mathrm{R}(A)$.
The operator A is called *monotone* if the following condition is fulfilled:

$$\langle y - x, y^* - x^* \rangle \geq \; 0 \;\text{ for every two pairs } (x, x^*), (y, y^*) \in \mathrm{Gr}(A).$$

Let us mention that if A is monotone, then its inverse A^{-1} is also monotone from X^* into X.
For an operator $A : X \rightrightarrows X^*$, denote by co$A$ (resp. by $\bar{A}$) the operator $(\mathrm{co}A)x := \mathrm{co}(Ax)$ (resp. $\bar{A}x := \overline{Ax}$), $x \in X$. Here co means convex hull, while the overbar has the meaning of the closure of a set with respect to the norm topology in X^*. Obviously, $\mathrm{Dom}(A) = \mathrm{Dom}(\bar{A}) = \mathrm{Dom}(\mathrm{co}A)$ and if A is monotone then so are $\bar{A}$ and co$A$.
A monotone operator $A : X \rightrightarrows X^*$ is said to be *maximal* if its graph is not contained properly in the graph of any other monotone operator from X to X^*. In other words, A is maximal, if and only if, whenever one has $\langle x - y, x^* - y^* \rangle \geq 0$ for every $(y, y^*) \in \mathrm{Gr}(A)$, it follows that $(x, x^*) \in \mathrm{Gr}(A)$. It is easily seen that, if A is a maximal monotone operator, then the operators $A, \bar{A}, \mathrm{co}A, \overline{\mathrm{co}}A$ coincide. Evidently, if A is maximal monotone then A^{-1} is also a maximal monotone operator between X^* and X. Finally, it is easily checked that if A is a maximal monotone operator then its graph $\mathrm{Gr}(A)$ is a closed subset with respect to the product of the norm topologies in $X \times X^*$.
Among the most important examples of maximal monotone operators are the subdifferentials of convex functions. Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be an extended real-valued proper lower semicontinuous convex function in X (proper means that the *domain* of f, dom $f := \{x \in X : f(x) < +\infty\}$ is nonempty). Given $\varepsilon \geq 0$, the *ε-subdifferential* of f is defined at $x \in \mathrm{dom}f$ by:

$$\partial_\varepsilon f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^* \rangle - \varepsilon \quad \text{for every } y \in X\},$$

and $\partial_\varepsilon f(x) := \emptyset$, if $x \notin \mathrm{dom}f$. It is known that $\mathrm{Dom}(\partial_\varepsilon f) = \mathrm{dom}f$ for every $\varepsilon > 0$. When $\varepsilon = 0$, $\partial_0 f$ is the *subdifferential* ∂f of f (the latter can be empty at some points of dom$f$). Obviously, one has $\partial f(x) \subset \partial_\varepsilon f(x)$ for every $x \in X$ and $\varepsilon > 0$.
The following result, which has been also a motivation for our investigations, was proved by Hiriart-Urruty and Phelps:

Theorem 2.1 (Hiriart-Urruty and Phelps [12]) *Let $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions. Then for every $x \in$ $\mathrm{dom} f \cap \mathrm{dom} g$ one has:*

$$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon f(x) + \partial_\varepsilon g(x)}.$$

The result is true in a general Banach space, in which case the closure in the right hand side is taken with respect to the weak star topology in X^*.

3 The concept of extended sum

The notion of approximate subdifferential has inspired the search of similar enlargements for monotone operators. The following one has been studied intensively in the last years:
Given a monotone operator $A : X \rightrightarrows X^*$ and $\varepsilon \geq 0$, the ε-*enlargement* of A is the operator $A^\varepsilon : X \rightrightarrows X^*$, determined by

$$A^\varepsilon x := \{x^* \in X : \langle y - x, y^* - x^* \rangle \geq -\varepsilon \quad \text{for any } (y, y^*) \in \mathrm{Gr}(A)\}.$$

This concept was observed by several authors (see e.g. [19]). But for the first time a detailed study was provided in [8] in finite dimensions, with applications to approximate solutions of variational inequalities, and in [9] with applications to finding a zero of a maximal monotone operator. Similar notions could be found in [18, 21, 22, 25, 34].
It is seen that A^ε is with closed convex images for any $\varepsilon \geq 0$. Due to the monotonicity of A, one has $Ax \subset A^\varepsilon x$ for every $x \in X$ and every $\varepsilon \geq 0$. I.e. A^ε is indeed an enlargement of A. If A and B are two monotone operators such that $A \subset B$ (equivalently $\mathrm{Gr}(A) \subset \mathrm{Gr}(B)$) then $B^\varepsilon \subset A^\varepsilon$ for every $\varepsilon \geq 0$! In the particular case when $A = \partial f$ one has $\partial_\varepsilon f \subset (\partial f)^\varepsilon$ and the inclusion can be strict (for instance for the function $f(x) = x^2$, $x \in \mathbf{R}$, cf. e.g. [19]). Let us mention that a Brøndsted-Rockafellar lemma is true for these enlargements (see Torralba [33] for the case of reflexive spaces (cf. also [9] where the Hilbert space setting is considered) and our paper [26] for a generalization outside these settings). For other properties of this notion we refer to [8, 9, 26].
Let $A, B : X \rightrightarrows X^*$ be two monotone operators. As usual the operator $A+B :$ $X \rightrightarrows X^*$ means the pointwise sum of A and B:

$$(A+B)x = Ax + Bx, \quad x \in X.$$

$A + B$ is a monotone operator with domain $\mathrm{Dom}(A + B) = \mathrm{Dom}A \cap \mathrm{Dom}B$. But it is well known that if A and B are maximal monotone operators, then, in general, their sum $A + B$ need not be a maximal monotone operator. A very well-known case of this phenomenon is the sum of the subdifferentials of two convex proper lower semicontinous functions which can be strictly less than the subdifferential of the sum of the functions. Other counterexamples for this, as

well as sufficient conditions for the maximality of the pointwise sum, could be found in the monographs of Phelps [24] and Simons [28].
As we pointed out, the above phenomenon motivated the study of possible generalized sums of monotone operators. One is the variational sum which will be considered in the next section. Another one was proposed in our paper [26] using the above enlargements:
The *extended sum* of two monotone operators $A, B : X \rightrightarrows X^*$ at the point $x \in X$ is defined in [26] by

$$A \underset{ext}{+} B(x) = \bigcap_{\varepsilon > 0} \overline{A^\varepsilon x + B^\varepsilon x}.$$

In the general case of an arbitrary Banach space, the closure on the right hand side is taken in [26] with respect to the weak star topology in X^* (which, of course, in our setting coincides with the norm closure, due to the reflexivity of the space and convexity of the images of the enlargements). Obviously, $A+B \subset \overline{A+B} \subset A \underset{ext}{+} B$ and hence, $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) \subset \mathrm{Dom}(A \underset{ext}{+} B)$. Moreover, the extended sum is commutative. As it is shown in [26], in several important cases the extended sum is a maximal monotone operator: e.g.:

Theorem 3.1 ([26], Theorem 3.1 and Corollary 3.2). *Let $\overline{A+B}$ (resp. $A+B$) be a maximal monotone operator. Then, $\overline{A+B} = A \underset{ext}{+} B$ (resp. $A+B = A \underset{ext}{+} B$).*

Moreover, the subdifferential of the sum of two convex proper lower semicontinuous functions is equal to the extended sum of their subdifferentials ([26], Theorem 3.3; see also the particular case of this result at end of this paper).

4 Variational sum of monotone operators

In this section we see how the notion of variational sum of maximal monotone operators introduced in the setting of Hilbert spaces by Attouch, Baillon and Théra in [3], can be extended to the case of reflexive Banach spaces, keeping its properties. We compare it then with the pointwise and extended sum.
For the extension of the variational sum to the more general setting we will follow the same scheme of approximations used in [3]. For this we need first to introduce the well-known *duality mapping* J between X and X^*, defined by:

$$Jx := \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

J is the subdifferential of the convex function $f(x) = (1/2)\|x\|^2, x \in X$ (see e.g. Phelps [24], Example 2.26), i.e. it is a maximal monotone operator with $\mathrm{Dom}(J) = X$. Remember that we consider norms in X and X^* which are Fréchet differentable away from the origin and locally uniformly routund, in particular with the Kadec-Klee property. In such a situation, the duality mapping J between X and X^* is one-to-one, surjective and norm-to-norm continuous.

Before giving the regularization scheme, let us recall a well-known result due to Rockafellar [27], Proposition 1 (which is also a generalization of the classical result of Minty for Hilbert spaces): in our setting of reflexive Banach spaces, with the norms we consider, if $A : X \rightrightarrows X^*$ is a maximal monotone operator then for any $\lambda > 0$ one has $\mathrm{R}(A + \lambda J) = X^*$ and $(A + \lambda J)^{-1}$ is a single-valued maximal monotone operator which is norm-to-weak continuous (the latter notion is termed also *demi-continuous*).

We proceed by introducing the elements of the regularization scheme. Given a maximal monotone operator $A : X \rightrightarrows X^*$ and a positive λ, there are two equivalent ways to define the well-known *Yosida regularization* (or *Yosida approximation*) of A of order λ: the first one is naturally motivated by the special case of subdifferentials of convex functions; one defines first the *resolvent* J_λ^A of A for $\lambda > 0$, as the operator from X into X determined by: for any $x \in X$, $J_\lambda^A x$ is the unique (by the result of Rockafellar above) solution x_λ of the inclusion:

$$0 \in J(x_\lambda - x) + \lambda A x_\lambda. \tag{4.1}$$

Then, the Yosida regularization $A_\lambda : X \rightrightarrows X^*$ of A is defined by

$$A_\lambda x := \frac{1}{\lambda} J(x - x_\lambda), \quad x \in X. \tag{4.2}$$

One easily sees that for each $\lambda > 0$

$$J_\lambda^A x = x - \lambda J^{-1} A_\lambda x \quad \text{for every } x \in X. \tag{4.3}$$

The second equivalent way to define the above regularization is purely analytical:

$$A_\lambda := (A^{-1} + \lambda J^{-1})^{-1}, \ \lambda > 0. \tag{4.4}$$

By the result of Rockafellar above, for any $\lambda > 0$, the Yosida regularization A_λ is an everywhere defined single-valued maximal monotone operator which is demi-continuous.

When X is a Hilbert space, then J is the identity and the above formulae reduce to $A_\lambda = (J - J_\lambda^A)/\lambda$ and $J_\lambda^A = (J + \lambda A)^{-1}$.

Observe that (4.1) and (4.2) above show that, in particular, for any $\lambda > 0$:

$$A_\lambda x \in A(J_\lambda^A x) \text{ for every } x \in X. \tag{4.5}$$

Let us list several well-known properties of the Yosida regularizations due to Brezis, Crandall and Pazy [6]. In what follows, given a maximal monotone operator $A : X \rightrightarrows X^*$ and $x \in \mathrm{Dom}(A)$, by $A^{\min}x$ we denote the unique element of Ax which has minimal norm in Ax (the existence of such an element is guaranteed by the reflexivity of X and the maximality of A (Ax is closed and convex), while its uniqueness follows by the fact that the norm in X^* is strictly convex).

Proposition 4.1 (see [6], Lemma 1.3) *Let A be a maximal monotone operator between the reflexive Banach space X and its dual X^* and the norm in X and the dual norm in X^* are locally uniformly rotund. Then*

(a) *for every* $\lambda > 0$ *the Yosida regularization* A_λ *maps bounded sets into bounded sets;*

(b) *for every* $\lambda > 0$ *and* $x \in \mathrm{Dom}(A)$ *we have* $\|A_\lambda x\| \leq \|A^{\min} x\|$*;*

(c) *for every* $x \in \mathrm{Dom}(A)$*,* $J_\lambda^A x$ *strongly converges to* x *as* $\lambda \downarrow 0$ *and* $A_\lambda x$ *strongly converges to* $A^{\min} x$ *as* $\lambda \downarrow 0$*.*

In fact, the second statement in (c) above is not proved in [6], Lemma 1.3, but can be derived from the asertions (d) and (e) of the same Lemma 1.3, using the fact that the norm in X^* satisfies the Kadec-Klee property. Indeed, in order to prove that $A_\lambda x$ strongly converges to $A^{\min} x$ as $\lambda \downarrow 0$, it is enough to show that any subsequence $A_{\lambda_n} x$ such that $\lambda_n \downarrow 0$ has a further subsequence which strongly converges to $A^{\min} x$. Let $A_{\lambda_n} x$ be such that $\lambda_n \downarrow 0$. By (b) above there exists a subsequence $A_{\lambda_{n_k}} x$ which converges weakly, say to y^*. By condition (e) of Lemma 3.1 from [6] $(x, y^*) \in A$ and hence, again by (b) above, $y^* = A^{\min} x$. Now, using (b) above and the (weak) lower semicontinuity of the norm in X^* we have

$$\|A^{\min} x\| \geq \limsup_k \|A_{\lambda_{n_k}} x\| \geq \liminf_k \|A_{\lambda_{n_k}} x\| \geq \|A^{\min} x\|.$$

Therefore, $\lim_k \|A_{\lambda_{n_k}} x\| = \|A^{\min} x\|$ and by the Kadec-Klee property we conclude that $A_{\lambda_{n_k}} x$ strongly converges to $A^{\min} x$.

Let now $\mathcal{I} := \{(\lambda, \mu) \in \mathbf{R}^2 : \lambda \geq 0, \mu \geq 0, \quad \lambda + \mu \neq 0\}$. Given two maximal monotone operators $A, B : X \rightrightarrows X^*$, the idea of the variational sum is to consider the operators of the type $A_\lambda + B_\mu$ for $(\lambda, \mu) \in \mathcal{I}$ and to take as a sum of A and B an appropriate limit of the above perturbations. Here for convention $A_0 = A$. Observe that, since $\lambda + \mu \neq 0$, $A_\lambda + B_\mu$ is always a maximal monotone operator due to a classical result of Browder [7]. Let $\mathcal{F}$ be the filter of all neighborhoods of the zero in $\mathcal{I}$. Then writing $\lim_{\mathcal{F}}$ we will have in mind the limit when $\lambda \to 0$, $\mu \to 0$, $(\lambda, \mu) \in \mathcal{I}$.

A natural idea related to convergence of operators, that has turned out to be useful in different investigations, is the idea of graph convergence (see Attouch [2]). This simply means that one identifies the operators with their graphs in $X \times X^*$ and considers on the latter the Painlevé-Kuratowski convergence determined by an appropriate convergence in the Cartesian product $X \times X^*$.

Given a family of operators $\{C_{\lambda,\mu} : (\lambda, \mu) \in \mathcal{I}\}$ between X and X^*, let us remind the definitions of lower and upper limit of this family in the sense of Painlevé-Kuratowski, when we consider in $X \times X^*$ the product topology generated by the strong topologies in X and X^*.

The lower limit of the family $\{C_{\lambda,\mu} : (\lambda, \mu) \in \mathcal{I}\}$ in the sense of Painlevé-Kuratowski is the following set:

$\|\cdot\| \times \|\cdot\| - \liminf_{\mathcal{F}} C_{\lambda,\mu} := \{(x, x^*) \in X \times X^* :$ for every $\|\cdot\| \times \|\cdot\|$-neighborhood $\mathcal{U}$ of (x, x^*) there exists $F \in \mathcal{F}$ such that $\mathcal{U} \cap \mathrm{Gr}(C_{\lambda,\mu}) \neq \emptyset$ for every $(\lambda, \mu) \in F\}$,

while its upper limit in the same sense is:

$\|\cdot\| \times \|\cdot\| - \limsup_{\mathcal{F}} C_{\lambda,\mu} := \{(x,x^*) \in X \times X^* :$ for every $\|\cdot\| \times \|\cdot\|$-neighborhood $\mathcal{U}$ of (x,x^*) and for every $F \in \mathcal{F}$ there exists $(\lambda,\mu) \in F$ such that $\mathcal{U} \cap \mathrm{Gr}(C_{\lambda,\mu}) \neq \emptyset\}$.

The family of monotone operators $\{C_{\lambda,\mu} : (\lambda,\mu) \in \mathcal{I}\}$ graph-converges to the monotone operator $C : X \Longrightarrow X^*$ if

$$\mathrm{Gr}(C) = \|\cdot\| \times \|\cdot\| - \liminf_{\mathcal{F}} C_{\lambda,\mu} = \|\cdot\| \times \|\cdot\| - \limsup_{\mathcal{F}} C_{\lambda,\mu}.$$

With slight abuse of notation we will omit in the above equality the symbol Gr and will write $C = \|\cdot\| \times \|\cdot\| - \lim_{\mathcal{F}} C_{\lambda,\mu}$.
It is easily seen that equivalent sequential definitions of the above notions are: $(x,x^*) \in \|\cdot\| \times \|\cdot\| - \liminf_{\mathcal{F}} C_{\lambda,\mu}$ exactly when for every sequence $\{(\lambda_n,\mu_n)\} \subset \mathcal{I}$ such that $\lambda_n,\mu_n \to 0$, there exists a sequence $(x_n,x_n^*) \in C_{\lambda_n,\mu_n}$ such that $(x_n,x_n^*) \to (x,x^*)$ strongly; $(x,x^*) \in \|\cdot\| \times \|\cdot\| - \limsup_{\mathcal{F}} C_{\lambda,\mu}$ iff there exists a sequence $\{(\lambda_n,\mu_n)\} \subset \mathcal{I}$ with $\lambda_n,\mu_n \to 0$ and such that there is a sequence $(x_n,x_n^*) \in C_{\lambda_n,\mu_n}$ for which $(x_n,x_n^*) \to (x,x^*)$ strongly.
Using the latter definitions, one can define $\|\cdot\| \times seq.-w-$(lower, upper) limit of a family of operators, when one considers on X^* the sequential weak convergence. In this case, one gets formally bigger sets as limits. But as we will see below, in the case of maximal monotone operators we have the same limits.
The next proposition is well-known (see e.g. [3]).

Proposition 4.2 *Let X be a reflexive Banach space and $C, \{C_{\lambda,\mu} : (\lambda,\mu) \in \mathcal{I}\}$ be maximal monotone operators between X and X^*. Then*

(a) $C = \|\cdot\| \times \|\cdot\| - \lim_{\mathcal{F}} C_{\lambda,\mu}$ *if and only if* $C \subset \|\cdot\| \times \|\cdot\| - \liminf_{\mathcal{F}} C_{\lambda,\mu}$;

(b) $C = \|\cdot\| \times seq.-w-\lim_{\mathcal{F}} C_{\lambda,\mu}$ *if and only if* $C \subset \|\cdot\| \times seq.-w-\liminf_{\mathcal{F}} C_{\lambda,\mu}$.

An obvious corollary from the above is:

Corollary 4.3 *Let X be a reflexive Banach space and $C, \{C_{\lambda,\mu} : (\lambda,\mu) \in \mathcal{I}\}$ be maximal monotone operators between X and X^*. Then $C = \|\cdot\| \times \|\cdot\| - \lim_{\mathcal{F}} C_{\lambda,\mu}$ implies $C = \|\cdot\| \times seq. - w - \lim_{\mathcal{F}} C_{\lambda,\mu}$.*

As it is seen from the following proposition, Yosida approximations of a given maximal monotone operator graph-converge to the operator.

Proposition 4.4 ([2]) *Let X be a reflexive Banach space and $A : X \Longrightarrow X^*$ be a maximal monotone operator. Then*

$$A = \|\cdot\| \times \|\cdot\| - \lim_{\lambda \downarrow 0} A_\lambda = \|\cdot\| \times seq. - w - \lim_{\lambda \downarrow 0} A_\lambda.$$

Now we pass to the definition of the variational sum. The following notion was introduced by Attouch, Baillon and Théra in [3] for the setting of Hilbert spaces. We give here its natural extension for the case of a reflexive Banach space.

Definition 4.5 *Let A and B be two maximal monotone operators in the reflexive Banach space X. The variational sum between A and B denoted by $A \underset{v}{+} B$ is the operator between X and X^* having the following graph:*

$$A \underset{v}{+} B := \|\cdot\| \times \|\cdot\| - \liminf_{\mathcal{F}} (A_\lambda + B_\mu).$$

As it was mentioned, an equivalent working definition is: $x^* \in (A \underset{v}{+} B)(x)$ exactly when for every sequence $\{(\lambda_n, \mu_n)\}_{n=1}^\infty \subset \mathcal{I}$ with $(\lambda_n, \mu_n) \to 0$ there are $x_n \in X$ and $x_n^* \in X^*$ such that $x_n \to x$, $x_n^* \to x^*$ and $x_n^* \in A_{\lambda_n} x_n + B_{\mu_n} x_n$ for every $n = 1, 2, \ldots$ It is seen that the definition of the variational sum at a certain point $x \in X$ takes into account the behaviour of the operators also at nearby points. This differs from the definition of extended sum above which involves the values of the approximations of the operators only at the point x. Let us prove some properties of the variational sum which are extensions of the similar ones established in Proposition 4.2, [3], in the setting of Hilbert spaces.

Proposition 4.6 *Let X be a reflexive Banach space and $A, B : X \rightrightarrows X^*$ be maximal monotone operators. Then*

(1) $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) \subset \mathrm{Dom}(A \underset{v}{+} B)$;

(2) $A \underset{v}{+} B$ *is a monotone operator;*

(3) *If $A \underset{v}{+} B$ is a maximal monotone operator then* $A \underset{v}{+} B = \|\cdot\| \times \|\cdot\| - \lim_{\mathcal{F}}(A_\lambda + B_\mu) = \|\cdot\| \times seq. - w - \lim_{\mathcal{F}}(A_\lambda + B_\mu)$;

(4) $A \underset{v}{+} B = B \underset{v}{+} A$.

Proof: (1) Let $x \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$. Then by Proposition 4.1 we have that $A_\lambda x$ $\|\cdot\|$-converges to $A^{\min} x$ as $\lambda \downarrow 0$ and $B_\mu x$ $\|\cdot\|$-converges to $B^{\min} x$ as $\mu \downarrow 0$. Hence $(x, A^{\min} x + B^{\min} x) \in \mathrm{Gr}(A \underset{v}{+} B)$. (3) follows by Proposition 4.2 and Corollary 4.3 and (4) is clear by the definition. As to (2), take $(x, x^*), (y, y^*) \in \mathrm{Gr}(A \underset{v}{+} B)$. Let $(\lambda_n, \mu_n) \in \mathcal{I}$ be a sequence so that $\lambda_n, \mu_n \to 0$ and $\lambda_n, \mu_n > 0$. By the definition of the variational sum for every n there are couples $(x_n, x_n^*) \in \mathrm{Gr}(A_{\lambda_n} + B_{\mu_n})$ and $(y_n, y_n^*) \in \mathrm{Gr}(A_{\lambda_n} + B_{\mu_n})$ such that $(x_n, x_n^*) \to (x, x^*)$ and $(y_n, y_n^*) \to (y, y^*)$ strongly. Having in mind this (and that $\lambda_n, \mu_n > 0$) we get

$$\begin{aligned}
\langle x - y, x^* - y^* \rangle &= \lim_n \langle x_n - y_n, x_n^* - y_n^* \rangle \\
&= \lim_n \langle x_n - y_n, A_{\lambda_n} x_n + B_{\mu_n} x_n - A_{\lambda_n} y_n - B_{\mu_n} y_n \rangle \\
&\geq 0,
\end{aligned}$$

the last inequality being true because of monotonicity of the Yosida approximations. The proof is completed. ■

Before passing to the relation between the variational sum and the extended one, we need an important auxiliary lemma.

Lemma 4.7 *Let A and B be two maximal monotone operators in the reflexive Banach space X such that* $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) \neq \emptyset$. *Suppose that* $(x, x^*) \in \mathrm{Gr}(A \underset{v}{+} B)$ *and that* $(x, x^*) = \|\cdot\| \times \|\cdot\| - \lim_{\lambda,\mu\downarrow 0}(x_{\lambda,\mu}, A_\lambda x_{\lambda,\mu} + B_\mu x_{\lambda,\mu})$. *Then* $\|\cdot\| - \lim_{\lambda,\mu\downarrow 0} \lambda A_\lambda x_{\lambda,\mu} = 0$ *and* $\|\cdot\| - \lim_{\lambda,\mu\downarrow 0} \mu B_\mu x_{\lambda,\mu} = 0$.

Proof: We will use a piece of argument from [3]. Let $y \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ and take $u^* \in Ay$ and $v^* \in By$. Let $\lambda, \mu > 0$ be fixed. Then using (4.5) and the monotonicity of A and B we have the following inequalities:

$$\langle y - J_\lambda^A x_{\lambda,\mu}, u^* - A_\lambda x_{\lambda,\mu}\rangle \geq 0$$

$$\langle y - J_\mu^B x_{\lambda,\mu}, v^* - B_\mu x_{\lambda,\mu}\rangle \geq 0.$$

Now by (4.3) we get

$$\langle y - x_{\lambda,\mu} + \lambda J^{-1}(A_\lambda x_{\lambda,\mu}), u^* - A_\lambda x_{\lambda,\mu}\rangle \geq 0$$

$$\langle y - x_{\lambda,\mu} + \mu J^{-1}(B_\mu x_{\lambda,\mu}), v^* - B_\mu x_{\lambda,\mu}\rangle \geq 0.$$

After adding these two inequalities we obtain:

$$\langle y - x_{\lambda,\mu}, u^* + v^* - (A_\lambda x_{\lambda,\mu} + B_\mu x_{\lambda,\mu})\rangle +$$
$$\langle \lambda J^{-1}(A_\lambda x_{\lambda,\mu}), u^* - A_\lambda x_{\lambda,\mu}\rangle + \langle \mu J^{-1}(B_\mu x_{\lambda,\mu}), v^* - B_\mu x_{\lambda,\mu}\rangle \geq 0.$$

Further, we use the definition of the duality mapping to get that:

$$\begin{aligned} &\langle y - x_{\lambda,\mu}, u^* + v^* - (A_\lambda x_{\lambda,\mu} + B_\mu x_{\lambda,\mu})\rangle \\ &\quad + \lambda\|u^*\|\,\|A_\lambda x_{\lambda,\mu}\| + \mu\|v^*\|\,\|B_\mu x_{\lambda,\mu}\| \\ &\geq \lambda\|A_\lambda x_{\lambda,\mu}\|^2 + \mu\|B_\mu x_{\lambda,\mu}\|^2. \end{aligned} \tag{4.6}$$

Observe that, for fixed y, u^*, v^*, when $\lambda, \mu \downarrow 0$, the first term on the left hand side of the above inequality is bounded because of the assumptions of the lemma. Hence one concludes from the last inequality that there is some constant $M > 0$ so that $\lambda^{1/2}\|A_\lambda x_{\lambda,\mu}\| < M$ and $\mu^{1/2}\|B_\mu x_{\lambda,\mu}\| < M$ for sufficiently small $\lambda, \mu > 0$, whence we deduce the assertion of the lemma. The proof is completed. ■

Remark 4.8 Let us stress, in connection with the proof of Theorem 4.12 below, that the inequality (4.6) above is true for every $y \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, every $u^* \in Ay$, $v^* \in By$ and $x_{\lambda,\mu}$, $\lambda, \mu > 0$, without supposing the convergence of $\{(x_{\lambda,\mu}, A_\lambda x_{\lambda,\mu} + B_\mu x_{\lambda,\mu})\}$.

Now we pass to the comparison of the variational sum with the usual pointwise sum and the extended sum.

Theorem 4.9 *Let X be a reflexive Banach space and $A, B : X \rightrightarrows X^*$ be two maximal monotone operators with* $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) \neq \emptyset$ *and such that $A \underset{v}{+} B$ is a maximal monotone operator. Then for every $x \in X$ we have*

$$(A \underset{ext}{+} B)(x) \subset (A \underset{v}{+} B)(x).$$

Proof: Let $x^* \in (A \underset{ext}{+} B)(x) = \bigcap_{\varepsilon>0} \overline{A^\varepsilon(x) + B^\varepsilon(x)}$ for some $x \in X$. We will show that

$$\langle x - y, x^* - y^* \rangle \geq 0 \text{ for every } (y, y^*) \in \mathrm{Gr}(A \underset{v}{+} B). \tag{4.7}$$

This, together with maximality of the variational sum, will imply that $x^* \in (A \underset{v}{+} B)(x)$.

To this end, take an arbitrary $(y, y^*) \in \mathrm{Gr}(A \underset{v}{+} B)$ and fix it. Let $\varepsilon > 0$ be an arbitrary fixed positive number. Since x^* belongs to the norm-closure of $A^\varepsilon(x) + B^\varepsilon(x)$ there exist $u_\varepsilon^* \in A^\varepsilon(x)$ and $v_\varepsilon^* \in B^\varepsilon(x)$ so that

$$\|x - y\| \, \|x^* - u_\varepsilon^* - v_\varepsilon^*\| \leq \varepsilon. \tag{4.8}$$

Fix these u_ε^* and v_ε^*. Further, for every $n = 1, 2, \ldots$, take $\lambda_n, \mu_n > 0$ so that $\lim_n \lambda_n = \lim_n \mu_n = 0$. Since $(y, y^*) \in \mathrm{Gr}(A \underset{v}{+} B)$ we get the existence of a sequence $\{(y_n, y_n^*)\}_{n=1}^\infty$ such that $y = \|\cdot\| - \lim_n y_n$, $y^* = \|\cdot\| - \lim_n y_n^*$ and $y_n^* = A_{\lambda_n} y_n + B_{\mu_n} y_n$. Let $M > 0$ be an upper bound of the norms of the sequence $\{y_n^*\}_{n=1}^\infty$ and n be so large that:

$$\begin{aligned} &|\langle x - y, y_n^* - y^* \rangle| < \varepsilon \\ &\|y - y_n\| < \frac{\varepsilon}{\|u_\varepsilon^* + v_\varepsilon^*\| + M}. \end{aligned} \tag{4.9}$$

Finally, using Lemma 4.7 we choose further n to be so large that

$$\begin{aligned} &\|u_\varepsilon^*\| \, \|\lambda_n A_{\lambda_n} y_n\| < \varepsilon \\ &\|v_\varepsilon^*\| \, \|\mu_n B_{\mu_n} y_n\| < \varepsilon. \end{aligned} \tag{4.10}$$

Now, having in mind (4.8) and (4.9), we obtain the following chain of inequalities:

$$\begin{aligned} \langle x - y, x^* - y^* \rangle &= \langle x - y, u_\varepsilon^* + v_\varepsilon^* - y^* \rangle + \langle x - y, x^* - u_\varepsilon^* - v_\varepsilon^* \rangle \\ &\geq \langle x - y, u_\varepsilon^* + v_\varepsilon^* - y_n^* \rangle + \langle x - y, y_n^* - y^* \rangle \\ &\quad - \|x - y\| \, \|x^* - u_\varepsilon^* - v_\varepsilon^*\| \\ &\geq \langle x - y_n, u_\varepsilon^* + v_\varepsilon^* - y_n^* \rangle \\ &\quad + \langle y_n - y, u_\varepsilon^* + v_\varepsilon^* - y_n^* \rangle - 2\varepsilon \\ &\geq \langle x - y_n, u_\varepsilon^* + v_\varepsilon^* - y_n^* \rangle \\ &\quad - \|y_n - y\| \left(\|u_\varepsilon^* + v_\varepsilon^*\| + M \right) - 2\varepsilon \\ &\geq \langle x - y_n, u_\varepsilon^* + v_\varepsilon^* - A_{\lambda_n} y_n - B_{\mu_n} y_n \rangle - 3\varepsilon. \end{aligned}$$

Next we use (4.3) to continue the above chain of inequalities as follows:

$$\begin{aligned} \langle x - y, x^* - y^* \rangle &\geq \langle x - y_n, u_\varepsilon^* - A_{\lambda_n} y_n \rangle + \langle x - y_n, v_\varepsilon^* - B_{\mu_n} y_n \rangle - 3\varepsilon \\ &= \langle x - J_{\lambda_n}^A y_n - \lambda_n J^{-1}(A_{\lambda_n} y_n), u_\varepsilon^* - A_{\lambda_n} y_n \rangle \\ &\quad + \langle x - J_{\mu_n}^B y_n - \mu_n J^{-1}(B_{\mu_n} y_n), v_\varepsilon^* - B_{\mu_n} y_n \rangle - 3\varepsilon \\ &= \langle x - J_{\lambda_n}^A y_n, u_\varepsilon^* - A_{\lambda_n} y_n \rangle + \langle x - J_\mu^B y_n, v_\varepsilon^* - B_{\lambda_n} y_n \rangle \\ &\quad - \lambda_n \langle J^{-1}(A_{\lambda_n} y_n), u_\varepsilon^* - A_{\lambda_n} y_n \rangle \\ &\quad - \mu_n \langle J^{-1}(B_{\mu_n} y_n), v_\varepsilon^* - B_{\mu_n} y_n \rangle - 3\varepsilon. \end{aligned}$$

Remember now that $u_\varepsilon^* \in A^\varepsilon(x)$ and $v_\varepsilon^* \in B^\varepsilon(x)$. This together with (4.5) show that the first two terms on the right hand side of the last equality in the above chain are greater or equal to $-\varepsilon$. Hence (using also the definition of J^{-1} and (4.10)):

$$\begin{aligned}\langle x-y, x^*-y^*\rangle &\geq -\lambda_n \langle J^{-1}(A_{\lambda_n} y_n), u_\varepsilon^* - A_{\lambda_n} y_n\rangle \\ &\quad -\mu_n \langle J^{-1}(B_{\mu_n} y_n), v_\varepsilon^* - B_{\mu_n} y_n\rangle - 5\varepsilon \\ &= \lambda_n \|A_{\lambda_n} y_n\|^2 - \lambda_n \langle J^{-1}(A_{\lambda_n} y_n), u_\varepsilon^*\rangle \\ &\quad +\mu_n \|B_{\mu_n} y_n\|^2 - \mu_n \langle J^{-1}(B_{\mu_n} y_n), v_\varepsilon^*\rangle - 5\varepsilon \\ &\geq -\lambda_n \|A_{\lambda_n} y_n\| \, \|u_\varepsilon^*\| - \mu_n \|B_{\mu_n} y_n\| \, \|v_\varepsilon^*\| - 5\varepsilon \\ &\geq -7\varepsilon.\end{aligned}$$

Consequently, we have proved that

$$\langle x-y, x^*-y^*\rangle \geq -7\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, we conclude that

$$\langle x-y, x^*-y^*\rangle \geq 0,$$

i.e. (4.7) is true. The proof is completed. ■

The following corollary is immediate.

Corollary 4.10 *Let A, B be two maximal monotone operators in the reflexive Banach space X such that $A \underset{v}{+} B$ is a maximal monotone operator. Then $A + B \subset A \underset{v}{+} B$.*

On the other hand, the inequality (4.7) was proved for arbitrary pairs from $A \underset{v}{+} B$ and $A \underset{ext}{+} B$. Therefore, we have the following theorem:

Theorem 4.11 *Let A, B be two maximal monotone operators in the reflexive Banach space X with $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) \neq \emptyset$ and such that $A \underset{ext}{+} B$ is a maximal monotone operator. Then $A \underset{v}{+} B \subset A \underset{ext}{+} B$.*

Finally, we prove that when $\overline{A+B}$ or $A+B$ is maximal, then all the three concepts of sum coincide. In particular, with this we extend for reflexive spaces Theorem 6.1 from [3]. Namely, we have:

Theorem 4.12 *Let X be a reflexive Banach space and $A, B : X \rightrightarrows X^*$ be two maximal monotone operators such that $\overline{A+B}$ is a maximal monotone operator. Then*

$$\overline{A+B} = A \underset{ext}{+} B = A \underset{v}{+} B.$$

Proof: Since $\overline{A+B}$ is maximal monotone, the conclusion of the theorem will follow by Theorem 4.9 if we prove the inclusion $\overline{A+B} \subset A \underset{v}{+} B$. For the latter, we will develop further an argument from the setting of Hilbert spaces used in [3].
Let $x^* \in \overline{A+B}(x)$ for some $x \in X$. Take a sequence $\{(\lambda_n, \mu_n)\}_{n=1}^{\infty} \subset \mathcal{I}$, so that $(\lambda_n, \mu_n) \to (0, 0)$. We will consider first the case $\lambda_n, \mu_n > 0$ for every $n = 1, 2, \ldots$
For any $n = 1, 2, \ldots$, let x_n be, by virtue of the result of Rockafellar above, the unique solution of the equality:

$$Jx + x^* = Jx_n + A_{\lambda_n} x_n + B_{\mu_n} x_n, \quad n = 1, 2, \ldots \tag{4.11}$$

Take arbitrary $y \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, $u^* \in Ay$ and $v^* \in By$. As it was mentioned in Remark 4.8, the inequality (4.6) is true if we put in the place of $x_{\lambda,\mu}$ the points x_n. Hence, using (4.6), (4.11) and the definition of the duality mapping we have for every $n = 1, 2, \ldots$

$$\begin{aligned} &\langle y - x_n, u^* + v^* - Jx - x^* \rangle + \langle y, Jx_n \rangle \\ &\quad + \lambda_n \|u^*\| \, \|A_{\lambda_n} x_n\| + \mu_n \|v^*\| \, \|B_{\mu_n} x_n\| \\ &\geq \lambda_n \|A_{\lambda_n} x_n\|^2 + \mu_n \|B_{\mu_n} x_n\|^2 + \|x_n\|^2. \end{aligned} \tag{4.12}$$

Using the last inequality, one can see that there exists a constant $M > 0$ so that $\|x_n\| \leq M$, $\sqrt{\lambda_n}\, \|A_{\lambda_n} x_n\| \leq M$ and $\sqrt{\mu_n}\, \|B_{\mu_n} x_n\| \leq M$ for every $n = 1, 2, \ldots$ In particular, $\lim_n \lambda_n \|A_{\lambda_n} x_n\| = \lim_n \mu_n \|B_{\mu_n} x_n\| = 0$.
Since $\{x_n\}$ is bounded, it has at least one weak cluster point, say $\bar{x}$. Now, since for every n, $\|x_n\|^2 = \langle x_n, Jx_n \rangle$ and since by the monotonicity of J we have $\langle x_n - y, Jx_n \rangle \geq \langle x_n - y, Jy \rangle$, we obtain from (4.12) that for every $n = 1, 2, \ldots$

$$\begin{aligned} &\langle y - x_n, u^* + v^* - Jx - x^* \rangle + \\ &\quad + \lambda_n \|u^*\| \, \|A_{\lambda_n} x_n\| + \mu_n \|v^*\| \, \|B_{\mu_n} x_n\| \\ &\geq \langle x_n - y, Jy \rangle. \end{aligned}$$

¿From here, taking a subsequence of $\{x_n\}$ which weakly converges to $\bar{x}$ and passing to the limit we get:

$$\langle y - \bar{x}, u^* + v^* - Jx - x^* \rangle \geq \langle \bar{x} - y, Jy \rangle,$$

or equivalently,

$$\langle y - \bar{x}, Jy + u^* + v^* - Jx - x^* \rangle \geq 0.$$

Since $u^* \in Ay$ and $v^* \in By$ were arbitrary, we conclude that

$$\langle y - \bar{x}, Jy + w^* - Jx - x^* \rangle \geq 0$$

for every $w^* \in \overline{A+B}(y)$. Now, remember that $\overline{A+B}$ was maximal, hence the same is true for $J + \overline{A+B}$. Therefore, since y was also arbitrary, the last inequality entails that $Jx + x^* \in J\bar{x} + \overline{A+B}(\bar{x})$. Finaly, since $x^* \in \overline{A+B}(x)$, the result of Rockafellar above implies that $\bar{x} = x$, from where we conclude that the whole sequence $\{x_n\}$ converges weakly to x.

Further, we show that $\|x_n\| \to \|x\|$. First, since J is the subdifferential of the function $(1/2)\|x\|^2$ we have for every $n = 1, 2, \ldots$

$$\langle x, Jx_n\rangle \leq \langle x_n, Jx_n\rangle + \frac{1}{2}\|x\|^2 - \frac{1}{2}\|x_n\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x_n\|^2.$$

Whence, using (4.12) with $y = x$ we easily get that $\limsup_n \|x_n\|^2 \leq \|x\|^2$. Therefore, using this and also the weak lower semicontinuity of the norm in X we obtain:

$$\|x\|^2 \leq \liminf_n \|x_n\|^2 \leq \limsup_n \|x_n\|^2 \leq \|x\|^2,$$

i.e. $\|x_n\|^2 \to \|x\|^2$, whence $\|x_n\| \to \|x\|$ and by the Kadec-Klee property we obtain that $\{x_n\}$ converges to x strongly. By (4.11) and the norm-to-norm continuity of J we deduce that $A_{\lambda_n}x_n + B_{\mu_n}x_n$ converges strongly to x^*.
In order to conclude that $(x, x^*) \in A \underset{v}{+} B$, it remains also to consider operators of the type $A_{\lambda_n} + B$ or $A + B_{\mu_n}$. But in this last case one uses the well-known results of Brezis, Crandall and Pazy [6], Theorem 2.1, to see that (x, x^*) is a norm-limit of a sequence from $A_{\lambda_n} + B$ or $A + B_{\mu_n}$. The proof is completed. ■

The following theorem is an immediate corollary from the above theorem:

Theorem 4.13 *Let X be a reflexive Banach space and $A, B : X \rightrightarrows X^*$ be two maximal monotone operators such that $A + B$ is a maximal monotone operator. Then*

$$A + B = A \underset{ext}{+} B = A \underset{v}{+} B.$$

5 Lower semicontinuous convex functions and sums of operators

In this section we see that the variational sum keeps another important property from the original setting of Hilbert spaces–namely, when the case of subdifferentials of convex functions is considered, we have that the subdifferential of the sum of two convex proper lower semicontinuous functions is equal to the variational sum of their subdifferentials. As a consequence we get that in the case of subdifferentials the extended and the variational sum are the same.
Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The well-known Moreau-Yosida regularization of f of order $\lambda > 0$ is given by the formula:

$$f_\lambda(x) := \inf_{y \in X}\{f(y) + \frac{1}{2\lambda}\|y - x\|^2\}, \quad x \in X.$$

It is also well-known that for a given $\lambda > 0$ the relation between the subdifferential of f_λ and the Yosida approximation of this subdifferential of order λ considered in the previous section is:

$$\partial(f_\lambda) = (\partial f)_\lambda. \tag{5.1}$$

Before giving the next result, let us remind the definition of *Mosco convergence* of convex functions ([20]): It is said that the sequence $\{f_n\}_{n=1}^{\infty}$ of proper lower semicontinuous convex functions in X Mosco-converges to f_0 if for every $x \in X$ the following two conditions are fulfilled:

(i) for every sequence $\{x_n\} \subset X$ which converges weakly to x we have $f(x) \leq \liminf_{n\to\infty} f_n(x_n)$;

(ii) there exists a sequence $\{x_n\} \subset X$ which strongly converges to x and such that $\limsup_{n\to\infty} f_n(x_n) \leq f(x)$.

The following fact was observed in [3], Theorem 7.2, for the setting of Hilbert spaces. We give its natural extension for the case of reflexive spaces we consider. In this last setting, the result has also been obtained directly by Jourani [15], using Ekeland variational principle. Other sequential formulae for the subdifferential of the sum of two convex functions can be found in the papers of Thibault [29, 32]. For similar results in the case of a general Banach space the reader is refered to the paper of Hiriart-Urruty and Phelps [12], the survey [13], the papers of Penot [23], Thibault [30, 31] and of the authors [26].

Theorem 5.1 *Let $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be two proper convex lower semicontinuous functions in the reflexive Banach space X such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. *Then*

$$\partial(f+g) = \partial f \underset{v}{+} \partial g.$$

Moreover,

$$\partial(f+g) = \|\cdot\| \times \|\cdot\| - \lim_{\mathcal{F}} (\partial f_\lambda + \partial g_\mu).$$

Proof: Since $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$ we see that $\partial(f+g)$ is a maximal monotone operator. The function f_λ for $\lambda > 0$ (or g_μ for $\mu > 0$) is everywhere defined and continuous. Hence, by the classical result of Moreau-Rockafellar $\partial f_\lambda + \partial g_\mu = \partial(f_\lambda + g_\mu)$ for every $(\lambda, \mu) \in \mathcal{I}$. Therefore, by (5.1) we conclude that

$$(\partial f)_\lambda + (\partial g)_\mu = \partial(f_\lambda + g_\mu) \quad \text{for all } (\lambda,\mu) \in \mathcal{I}. \tag{5.2}$$

On the other hand, by a result from [5], Theorem 3.20, we deduce that the family $\{f_\lambda + g_\mu : (\lambda, \mu) \in \mathcal{I}\}$ Mosco converges to $f + g$ as (λ, μ) converges to $(0, 0)$. Now, an already classical result of Attouch [1] allows to derive from the latter that $\|\cdot\| \times \|\cdot\| - \lim_{\mathcal{F}} \partial(f_\lambda + g_\mu) = \partial(f + g)$. It remains to take into account (5.2) above and the definition of the variational sum. ■

The following is an immediate corollary from the above theorem, Theorem 4.9 and the formula of Hiriart-Urruty and Phelps (Theorem 2.1). It shows that in the case of subdifferentials of convex functions the variational and the extended sum coincide.

Corollary 5.2 *Let $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be two proper convex and lower semicontinuous functions in the reflexive Banach space X such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. *Then:*

$$\partial(f+g) = \partial f \underset{v}{+} \partial g = \partial f \underset{ext}{+} \partial g.$$

In fact, the subdifferential of the sum of two convex proper lower semicontinuous functions is equal to the extended sum of their subdifferentials in the more general setting of an arbitrary Banach space as shown in our paper [26].

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