Enlargements and sums of monotone operators*

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Abstract

In this paper we study two important notions related to monotone operators. One is the concept of enlargement of a given monotone operator which has turned out to be a useful tool in the analysis of approximate solutions to problems involving monotone operators. The second one is the notion of sum of monotone operators. For the latter we introduce and study a kind of extended sum of two monotone operators, which, in several important cases, turns out to be a maximal monotone operator.

Key words: monotone operators, sums of operators, enlargements, subdifferentials

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1 Introduction

In the recent years two notions related to monotone (in general set-valued) operators have turned out to be very useful in the study of various problems involving such operators. The first one, which is inspired by the notion of ε-subdifferential of a convex function, is the concept of enlargement of a given operator (see e.g. [BuISv, BuSSv, MaT]). It allows to make a quantitative analysis in different problems involving monotone operators, like for example variational inequalities, inclusions etc. The second notion is the one

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of generalized sum of two maximal monotone operators (i.e. monotone operators whose graphs are maximal elements, with respect to the inclusion order, among the graphs of all monotone operators). It is well-known that the usual pointwise sum of two maximal monotone operators is not, in general, a maximal monotone operator. A typical example for the latter is the sum of the subdifferentials of two convex proper lower semicontinuous functions which may differ from the subdifferential of their sum. There are also other examples (i.e. in partial differential equations, see [ABT1, ABT2]), in which the resolution of a problem leads to an operator which appears to be larger than the sum of two maximal monotone operators. This motivated different authors to look for a generalized notion of sum of two operators, like e.g. the variational sum introduced by Attouch, Baillon and Théra in [ABT1] in the setting of Hilbert spaces, the parallel sum (see the study of electrical networks based on this notion in [K]), or sum based on the Trotter-Lie formula (see [Lap]).

This article is devoted to the study of the two notions mentioned above. For the notion of enlargement we consider a concept intensively studied recently in [BuISv, BuSSv], while for the notion of sum we will introduce an *extended sum* generated by the enlargements of the operators. The comparison of the latter concept with the variational sum of Attouch, Baillon and Théra, in the particular case of a reflexive Banach space, is treated in [RT2] (the announcement of the results is in [RT1]). Related to this comparison study, here we only mention the following: the extended sum is defined in this paper in a general Banach space while it seems the variational sum can be successfully extended only to the setting of reflexive Banach spaces (as shown in [RT2]). This is so since the variational sum between two maximal monotone operators is defined as an appropriate limit of the pointwise sum of certain (Yosida) approximations of the initial operators and these approximations are well-defined only in reflexive Banach spaces. The second remark concerns the relation between the extended and variational sum in reflexive Banach spaces (see again [RT2]): the maximality of one of them entails that it contains the other and vice versa; The two notions coincide, for example, in the case of maximality of the pointwise sum, and in the case of subdifferentials of convex functions (related to the latter, see also Theorem 4.4 below which shows that, without any qualification conditions and in any Banach space, the extended sum of the subdifferentials of two convex functions is equal to the subdifferential of their sum).

The paper is organized as follows. Section 2 provides some necessary preliminaries. In Section 3 we present and study the concept of ε-enlargement of a given maximal monotone operator. We show how, in the case of open domain of a given monotone operator A, one can obtain the unique maximal monotone operator (in the same domain) containing A by the enlargements of A. We establish further, for the class of monotone operators of type (D) introduced by Gossez [Go1], a Brøndsted-Rockafellar type theorem which holds also in non reflexive Banach spaces. Doing so we extend a previous unpublished result of Torralba [To] proved in reflexive Banach spaces. In the final Section 4 we introduce the notion of extended sum and study the relationship with the pointwise sum of maximal monotone operators. It is proved in particular, that when the pointwise sum of two monotone operators is a maximal monotone operator, the latter coincides with the extended sum of the two operators. Finally, we show that the subdifferential of the sum of two proper convex lower semicontinuous functions is exactly the extended sum of their subdifferentials.

2 Some preliminaries

Let X be a real Banach space with continuous dual X^*. The norm in X and X^* will be denoted by $||\cdot||$, w and w^* will stand for the weak and weak star topology in X and X^* correspondingly. The symbol $\langle\cdot,\cdot\rangle$ will be used for the usual pairing between X and X^*.

Let A be (in general, set-valued) operator between X and X^*. The *graph* of A will be denoted by

$$\mathrm{Gr}(A) := \{(x, x^*) \in X \times X^* : x^* \in Ax\}$$

and the sets

$$\mathrm{Dom}(A) := \{x \in X : \ Ax \neq \emptyset\}$$

and

$$\mathrm{R}(A) := \bigcup\{Ax : x \in \mathrm{Dom}(A)\}$$

stand as usual for the *domain* and the *range* of A respectively. The inverse operator of $A : X \to X^*$ will be designated by $A^{-1} : X^* \to X$. It is defined by:

$$A^{-1}x^* := \{x \in X : x^* \in Ax\}, \ x^* \in X^*,$$

and obviously has as domain the range of A.

Recall that the operator A is called *monotone* if the following condition is fulfilled:

$$\langle y - x, y^* - x^*\rangle \geq \ 0 \ \text{ for every two couples } (x, x^*), (y, y^*) \in \mathrm{Gr}(A).$$

Observe that if A is monotone, then so is its inverse A^{-1}.

Given an operator $A : X \to X^*$, by $\mathrm{co}A$, $\bar{A}$ and $\tilde{A}$ we will denote the following operators:

$$(\mathrm{co}A)x := \mathrm{co}(Ax), \ x \in X,$$

$$\bar{A}x := \overline{Ax}^{w^*}, x \ \in X,$$

and

$$\tilde{A} \text{ is such that } \mathrm{Gr}(\tilde{A}) = \overline{\mathrm{Gr}(A)}^{||\cdot||\times w^*}.$$

In the operations above co means convex hull while given $D \subset X^*$ and $H \subset X \times X^*$ the notations $\overline{D}^{w^*}$ and $\overline{H}^{||\cdot||\times w^*}$ are the closures of the sets D and H with respect to the weak star topology in X^* and with respect to the product of the norm and weak star topology in $X \times X^*$ correspondingly.

Obviously for the domains of the above operators we have $\mathrm{Dom}(\mathrm{co}A) = \mathrm{Dom}(\bar{A}) = \mathrm{Dom}(A) \subset \mathrm{Dom}(\tilde{A})$, while for their graphs it is clear that $\mathrm{Gr}(A) \subset \mathrm{Gr}(\bar{A}) \subset \mathrm{Gr}(\tilde{A})$ and $\mathrm{Gr}(A) \subset \mathrm{Gr}(\mathrm{co}A) \subset \mathrm{Gr}(\overline{\mathrm{co}}A) \subset \mathrm{Gr}(\tilde{\mathrm{co}}A)$.

The following remark contains well-known facts, which will be used in the sequel.

Remark 2.1

- if A is monotone then so are $\bar{A}$, co$A$ and $\overline{\text{co}}A$;
- if A is monotone, then so is $\tilde{A}$, provided A is locally bounded (cf. e.g. [DrLa, Ph1]).

Recall that A is *locally bounded* if for each point x from the norm-closure of Dom(A) there is a neighborhood U of x such that $A(U)$ is a norm-bounded subset in X^*.

A monotone operator $A : X \to X^*$ is said to be *maximal* if its graph is a maximal element in $X \times X^*$, with respect to the inclusion order, among the graphs of all monotone operators between X and X^*. I.e. A is maximal if the graph of A is not contained as a proper subset in the graph of any other monotone operator from X to X^*. Equivalently, A is maximal, if whenever one has $\langle x-y, x^*-y^*\rangle \geq 0$ for every $(y, y^*) \in \mathrm{Gr}(A)$, it follows that $(x, x^*) \in \mathrm{Gr}(A)$.

It is clear from the above remarks that if A is a maximal monotone operator, then the operators A, $\bar{A}, \mathrm{co}A, \overline{\mathrm{co}}A$ coincide, and if A is maximal monotone and locally bounded, then A coincides also with $\tilde{A}$ and $\tilde{\mathrm{co}}A$. Let us also mention the obvious fact that if A is maximal monotone then A^{-1} is also a maximal monotone operator between X^* and X.

A typical example of a maximal monotone operator is the subdifferential of a proper lower semicontinuous convex function $f : X \to \mathbf{R} \cup \{+\infty\}$. The term *proper* f means as usual that the set dom $f := \{x \in X : f(x) < +\infty\}$ (which is the *domain* of f) is nonempty. For a given $\varepsilon \geq 0$ the *ε-subdifferential* of f is defined as follows:

$$\partial_\varepsilon f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^*\rangle - \varepsilon \quad \text{for every } y \in X\},$$

if $x \in \mathrm{dom}f$, and $\partial_\varepsilon f(x) := \emptyset$, if $x \notin \mathrm{dom}f$. For every $\varepsilon > 0$, $\partial_\varepsilon f$ is always non-empty valued at the points of dom$f$, i.e. $\mathrm{Dom}\partial_\varepsilon f = \mathrm{dom}f$. When $\varepsilon = 0$, $\partial_0 f$ is exactly the *subdifferential* ∂f of f (the latter can be empty at some points of dom$f$).

Finally, we formulate a result due to Hiriart-Urruty and Phelps [HUPh] which we will need later and which was one of the starting points for our study:

Theorem 2.2 (Hiriart-Urruty and Phelps [HUPh]) *Let $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions. Then for every $x \in \mathrm{dom}f \cap \mathrm{dom}g$ one has:*

$$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon f(x) + \partial_\varepsilon g(x)}^{w^*}.$$

The sum under the bar in the right hand side of the above equality is the usual Minkowski pointwise sum of sets. Various similar formulas (also for other operations on functions) can be found in the survey [HUMSV].

3 Enlargements of monotone operators

Let A be a given monotone operator acting between X and X^*. Given $\varepsilon \geq 0$, by an *ε-enlargement* of the operator A we mean the operator $A^\varepsilon : X \to X^*$ defined by

(3.1) $$A^\varepsilon x := \{x^* \in X : \langle y - x, y^* - x^*\rangle \geq -\varepsilon \quad \text{for any } (y, y^*) \in \mathrm{Gr}(A)\}.$$

This notion was mentioned in [MaT] but not studied. Systematically, the above concept has been studied in finite dimensions in [BuISv] with the purpose of examining approximate solutions of variational inequalities, and in [BuSSv] in connection with finding a zero of a maximal monotone operator. Another type of enlargement of a monotone operator in Hilbert spaces is considered in [Ni]. Operators satisfying a similar condition as above have been investigated in [Ve]. Similar ideas for enlargements were used earlier in [LP, R] to define approximate solutions of variational inequalities determined by monotone operators.

It is seen from (3.1) that the operator A^ε is always with convex and w^*-closed images for any $\varepsilon \geq 0$ and that indeed, due to the monotonicity of A, it is an enlargement of A, i.e. $Ax \subset A^\varepsilon x$ for every $\varepsilon \geq 0$ and $x \in X$. For each $x \in X$ it is true also that $A^{\varepsilon_1}x \subset A^{\varepsilon_2}x$ provided $0 \leq \varepsilon_1 \leq \varepsilon_2$. But observe, in contrast of what is expected, that if A and B are two monotone operators such that $A \subset B$ (equivalently $\mathrm{Gr}(A) \subset \mathrm{Gr}(B)$) then $B^\varepsilon \subset A^\varepsilon$ for every $\varepsilon \geq 0$.

The idea of the above enlargement is to approximate the initial operator and clearly is inspired by the notion of ε-subdifferential of a proper lower semicontinuous convex function $f : X \to \mathbf{R} \cup \{+\infty\}$. In the case when $A = \partial f$ the enlargement given in (3.1) is larger than the ε-subdifferential, i.e. $\partial_\varepsilon f \subset (\partial f)^\varepsilon$, and there are examples showing that this inclusion can be strict ([MaT], i.e. $X = \mathbf{R}, f(x) = x^2$).

The natural question that appears is: given A, how good an approximation of A is the operator A^ε, $\varepsilon > 0$. We will see at the end of this section that a Brøndsted-Rockafellar type theorem is true for a class of operators, extending a previous unpublished result of Torralba [To], proved in the setting of reflexive Banach spaces. Before that we need some properties of the enlargements.

Given a monotone operator $A : X \to X^*$, it is readily seen that the operator $A^0 : X \to X^*$ (i.e. the enlargement in the partial case $\varepsilon = 0$) is obtained by intersecting all A^ε, i.e.

$$A^0 x := \bigcap \{A^\varepsilon x : \varepsilon > 0\}, \quad x \in X.$$

As we have remarked $A \subset A^0$. Recall that (see e.g. [Ph2]) a pair $(x, x^*) \in X \times X^*$ is said to be *monotonically related to* $\mathrm{Gr}(A)$ if $\langle y - x, y^* - x^* \rangle \geq 0$ for every couple $(y, y^*) \in \mathrm{Gr}(A)$. It is seen that $\mathrm{Gr}(A^0)$ is exactly the set of all pairs in $X \times X^*$ which are monotonically related to $\mathrm{Gr}(A)$. The operator A^0 need not be monotone (see below). But if A is maximal monotone the following proposition is immediate:

Proposition 3.1 *Let A be a maximal monotone operator. Then $A = A^0$.*

Hence we have the following corollary:

Corollary 3.2 *Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$\partial f = \bigcap_{\varepsilon > 0} (\partial f)^\varepsilon.$$

As we mentioned, in general, A^0 need not be monotone. But there is a class of monotone operators A for which A^0 is monotone (moreover, in this case A^0 is also maximal monotone). These are the so-called monotone operators of type (D) which were introduced by Gossez [Go1] for the purpose of extending to non-reflexive spaces some of the important properties of the maximal monotone operators in reflexive Banach spaces.

Identifying the Banach space X with its canonical embedding $\hat{X}$ in the second dual X^{**} and having a monotone operator $A : X \to X^*$, let $\hat{A}$ denote the operator A considered as a mapping from X^{**} into X^*, i.e. $\mathrm{Gr}(\hat{A}) = \{(\hat{x}, x^*) : (x, x^*) \in \mathrm{Gr}(A)\}$. A is called of *type (D)* ([Go1]) if for every couple $(x^{**}, x^*) \in X^{**} \times X^*$ which is monotonically related to $\mathrm{Gr}(\hat{A})$ there exists a net $\{(x_\alpha, x^*_\alpha)\} \subset \mathrm{Gr}(A)$ such that $\hat{x}_\alpha \to x^{**}$ in the weak star topology in X^{**}, $\{x_\alpha\}$ is bounded and $x^*_\alpha \to x^*$ in the norm. It can be verified (see e.g. [Ph2]) that if A is monotone of type (D) then the operator $\hat{A}^0 : X^{**} \to X^*$, which (as we have seen) has as a graph all monotonically related to $\mathrm{Gr}(\hat{A})$ pairs in $X^{**} \times X^*$, is maximal monotone (hence the same is true for A^0). But in general, $\hat{A}^0$ need not be monotone ([Go2]). Obviously, in reflexive Banach spaces (because of $\hat{X} = X^{**}$) the class of maximal monotone operators coincides with the class of maximal monotone operators of type (D). Finally, the subdifferential ∂f of every proper convex lower semicontinuous function f in X is a maximal monotone operator of type (D) ([Go1]).

Further, we investigate what are the relations between the enlargements of a given monotone operator and the enlargements of its natural extensions considered in the previous section.

Proposition 3.3 *Let $A : X \to X^*$ be monotone. Then for every $\varepsilon \geq 0$*

(i) $A^\varepsilon = \bar{A}^\varepsilon$;

(ii) *if A is locally bounded, then $A^\varepsilon = \tilde{A}^\varepsilon$;*

(iii) *if A is monotone of type (D) and the symbol $\hat{A}^\varepsilon$ denotes the ε-enlargement of A, when the latter is considered as a mapping from X^{**} to X^*, then $(\hat{A}^0)^\varepsilon = \hat{A}^\varepsilon$.*

Proof: Fix some $\varepsilon \geq 0$. By the remarks after the definition of ε-enlargements $\bar{A}^\varepsilon \subset A^\varepsilon$ and $\tilde{A}^\varepsilon \subset A^\varepsilon$. Conversely, suppose that $x^* \in A^\varepsilon x$ for some $x \in X$. Then

$$\langle y - x, y^* - x^* \rangle \geq -\varepsilon \;\text{ for any } (y, y^*) \in \mathrm{Gr}(A). \tag{3.2}$$

Take $(z, z^*) \in \mathrm{Gr}(\bar{A})$. Then $z^* \in \bar{A}z$ and consequently $z^* = w^* - \lim_\alpha z^*_\alpha$, where $\{z^*_\alpha\}$ is some net in Az. Since by (3.2)

$$\langle z - x, z^*_\alpha - x^* \rangle \geq -\varepsilon \;\text{ for any } \alpha,$$

we get by taking the limit that

$$\langle z - x, z^* - x^* \rangle \geq -\varepsilon.$$

Hence, $x^* \in \bar{A}^\varepsilon x$.

To finish the proof of (ii) suppose further that A is locally bounded. Let $(z, z^*) \in \mathrm{Gr}(\tilde{A})$. Then $(z, z^*) = ||\cdot|| \times w^* - \lim_\alpha (z_\alpha, z^*_\alpha)$, where $\{(z_\alpha, z^*_\alpha)\}$ is a net from $\mathrm{Gr}(A)$. By the choice of (x, x^*) and (3.2) we have

$$\langle z_\alpha - x, z^*_\alpha - x^* \rangle \geq -\varepsilon \quad \text{for any } \alpha.$$

Therefore,

$$\begin{aligned} \langle z - x, z^* - x^* \rangle &= \langle z_\alpha - x, z^*_\alpha - x^* \rangle + \langle z - z_\alpha, z^*_\alpha - x^* \rangle + \langle z - x, z^* - z^*_\alpha \rangle \\ &\geq -\varepsilon + \langle z - z_\alpha, z^*_\alpha - x^* \rangle + \langle z - x, z^* - z^*_\alpha \rangle \end{aligned}$$

Now using the fact that A is locally bounded we see that z^*_α are norm-bounded for α large enough, whence passing to the limit we get

$$\langle z - x, z^* - x^* \rangle \geq -\varepsilon.$$

Therefore, $x^* \in \tilde{A}^\varepsilon x$.

The proof of (iii) follows exactly the same pattern as the proof of (ii). The proof is complete. ■

Similarly to Proposition 3.3 we have

Proposition 3.4 *Let A be a monotone operator between X and X^*. Then $A^\varepsilon = (\mathrm{co}A)^\varepsilon$ for every $\varepsilon \geq 0$.*

Proof: Let $\varepsilon \geq 0$. Since $(\mathrm{co}A)^\varepsilon \subset A^\varepsilon$ is true, we will show the inverse inclusion. Let $x^* \in A^\varepsilon x$ for some $x \in X$ (observe that the couple (x, x^*) satisfies (3.2)). Take $(z, z^*) \in \mathrm{Gr}(\mathrm{co}A)$. Then $z^* \in (\mathrm{co}A)z = \mathrm{co}(Az)$ and consequently $z^* = \sum_{i=1}^n t_i z^*_i$ for some $t_i \geq 0$, $\sum_{i=1}^n t_i = 1$ and some $z^*_i \in Az$. Then we get (using also (3.2)) that

$$\begin{aligned} \langle z - x, z^* - x^* \rangle &= \langle z - x, \textstyle\sum_{i=1}^n t_i z^*_i - x^* \rangle \\ &= \textstyle\sum_{i=1}^n t_i \langle z - x, z^*_i - x^* \rangle \\ &\geq \textstyle\sum_{i=1}^n t_i(-\varepsilon) = -\varepsilon, \end{aligned}$$

which shows that $x^* \in (\mathrm{co}A)^\varepsilon(x)$. ■

The following is an immediate corollary from the previous propositions and the fact that if A is locally bounded then so is $\mathrm{co}A$.

Corollary 3.5 *Let $A : X \to X^*$ be a monotone operator. Then for every $\varepsilon \geq 0$ one has:*

(i) $A^\varepsilon = \bar{A}^\varepsilon = (\mathrm{co}A)^\varepsilon = (\overline{\mathrm{co}}A)^\varepsilon$;

(ii) *if A is locally bounded, then $A^\varepsilon = \tilde{A}^\varepsilon = (\tilde{\mathrm{co}}A)^\varepsilon = [\overline{\mathrm{co}}(\tilde{A})]^\varepsilon$.*

When the domain of A is open, the ε-enlargements can be used to obtain the unique maximal monotone operator in the domain which contains A. Namely, we have

Corollary 3.6 *Let A be a monotone operator between X and X^* such that* $\mathrm{Dom}(A)$ *is an open set in X. Then the restriction $A^0|_{\mathrm{Dom}(A)}$ is the unique maximal monotone operator in* $\mathrm{Dom}(A)$ *containing A. Moreover, $A^0|_{\mathrm{Dom}(A)} = \overline{\mathrm{co}}(\tilde{A}|_{\mathrm{Dom}(A)})$.*

Proof: It is a classical result of Rockafellar [Ro1] that every monotone operator $A : X \to X^*$ is locally bounded provided $\mathrm{Dom}(A)$ is open. By a result of Drewnowski and Labuda [DrLa], Theorem 3.7, when $\mathrm{Dom}(A)$ is open, the operator $\overline{\mathrm{co}}(\tilde{A}|_{\mathrm{Dom}(A)})$ is the unique maximal monotone operator in $\mathrm{Dom}(A)$ containing A. On the other hand, by the previous corollary, $A^0|_{\mathrm{Dom}(A)} = \bigcap_{\varepsilon>0} [\overline{\mathrm{co}}(\tilde{A}|_{\mathrm{Dom}(A)})]^{\varepsilon} = \overline{\mathrm{co}}(\tilde{A}|_{\mathrm{Dom}(A)})$, where the latter equality follows from the maximality of $\overline{\mathrm{co}}(\tilde{A}|_{\mathrm{Dom}(A)})$ in $\mathrm{Dom}(A)$ and Proposition 3.1. ∎

We end this section by showing that a Brøndsted-Rockafellar type theorem is true in general Banach spaces for the maximal monotone operators of type (D). Before that let us introduce a well-known object: denote by J the usual duality mapping between X and X^* defined by:

$$Jx := \{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \quad x \in X.$$

This mapping is in fact the subdifferential of the convex function $f(x) = (1/2)||x||^2, x \in X$ (see e.g. Phelps [Ph1], Example 2.26). Hence it is a maximal monotone operator. The domain of J is the whole space X. The duality mapping associated to the dual space X^* (with values in X^{**}) will be denoted by J^*.

In what follows, remember that for an operator A between X and X^*, $\hat{A}^0$ is the operator between X^{**} and X^* having as a graph all pairs in $X^{**} \times X^*$ which are monotonically related to $\hat{A}$. The following result is due to Gossez and is a further generalization of the classical result of Minty: *Let X be a Banach space and $A : X \to X^*$ be a maximal monotone operator of type (D). Then for every $\lambda > 0$ we have* $\mathrm{R}(\hat{A}^0 + \lambda(J^*)^{-1}) = X^*$. As it is well-known the same result for reflexive spaces is due to Rockafellar [Ro2].

Now we are ready to prove the following type of Brøndsted-Rockafellar theorem for ε-enlargements which is true in a general Banach space. When X is a reflexive Banach space, this result is proved by Torralba in his PhD thesis [To], Proposition 6.17 (c.f. also [BuSSv] where the setting of Hilbert spaces is treated). In the theorem below we have an approximation (in norm) of a couple from the enlargement of a given operator A by a couple from the graph of the operator $\hat{A}^0$ of the monotonically related (to $\hat{A}$) points in $X^{**} \times X^*$:

Theorem 3.7 *Let X be a Banach space and $A : X \to X^*$ be a maximal monotone operator of type (D). Then for every $\varepsilon > 0$, for every $(x, x^*) \in \mathrm{Gr}(A^{\varepsilon})$ and every $\lambda > 0$ there exists $(\bar{x}^{**}, \bar{x}^*) \in \mathrm{Gr}(\hat{A}^0)$ such that $||\bar{x}^{**} - \hat{x}|| \leq \lambda$ and $||\bar{x}^* - x^*|| \leq \varepsilon/\lambda$.*

Proof: The proof follows the scheme based on the generalization of the Minty theorem for monotone operators of type (D) formulated above.

Let $\varepsilon > 0$, $(x, x^*) \in \mathrm{Gr}(A^{\varepsilon})$ and $\lambda > 0$ be fixed. It is easily verified that the operator $B(\cdot) := A(\cdot + x)$ is also a maximal monotone operator of type (D) such that $\hat{B}^0(\cdot) =$

$\hat{A}^0(\cdot + \hat{x})$. Take $t = \varepsilon/\lambda^2$. By the Gossez generalization of the Minty theorem we have $\mathrm{R}(\hat{A}^0(\cdot + \hat{x}) + t(J^*)^{-1}) = X^*$. Hence, there exist $\bar{x}^{**} \in X^{**}$, $\bar{x}^* \in \hat{A}^0\bar{x}^{**}$ and $y^* \in (J^*)^{-1}(\bar{x}^{**} - \hat{x})$ so that

$$x^* = \bar{x}^* + ty^*. \tag{3.3}$$

By Proposition 3.3 (iii), $(\hat{x}, x^*) \in \mathrm{Gr}((\hat{A}^0)^\varepsilon)$. Using this fact, the definition of J^* and (3.3) we obtain:

$$-t||\bar{x}^{**} - \hat{x}||^2 = -\langle ty^*, \bar{x}^{**} - \hat{x}\rangle = \langle \bar{x}^* - x^*, \bar{x}^{**} - \hat{x}\rangle \geq -\varepsilon.$$

Since $t = \varepsilon/\lambda^2$ we conclude that $||\bar{x}^{**} - \hat{x}|| \leq \lambda$. On the other hand, by this inequality, the definition of J^* and (3.3) we have

$$||\bar{x}^* - x^*|| = t||y^*|| = t||\bar{x}^{**} - \hat{x}|| \leq t\lambda = \varepsilon/\lambda.$$

The proof is complete. ■

In the particular case of a reflexive Banach space one has $\hat{A}^0 = \hat{A}$ and hence the couple $(\bar{x}^{**}, \bar{x}^*)$ which approximates the original one is from the graph of the operator A. This is the result of Torralba mentioned above.

In the general case (of an arbitrary Banach space), if we want to have some approximation of a given pair from the enlargement of A by a pair from the graph of the very operator A, we have the following corollary in which the approximation in one of the coordinates is weaker than the norm-approximation:

Corollary 3.8 *Let A be a maximal monotone operator of type (D) between the Banach space X and its dual X^*. Then for every $\varepsilon > 0$ and for every $(x, x^*) \in \mathrm{Gr}(A^\varepsilon)$ there exists $(x_\varepsilon, x_\varepsilon^*) \in \mathrm{Gr}(A)$ such that:*

(i) $||x_\varepsilon^* - x^*|| \leq 2\sqrt{\varepsilon}$; *and*

(ii) $|\langle x_\varepsilon - x, x_\varepsilon^* - x^*\rangle| \leq 2\varepsilon$.

Proof: Fix some $\varepsilon > 0$ and $(x, x^*) \in \mathrm{Gr}(A^\varepsilon)$. By the theorem above (with $\lambda = \sqrt{\varepsilon}$) there is a couple $(\bar{x}^{**}, \bar{x}^*) \in \mathrm{Gr}(\hat{A}^0)$ such that:

(1) $||\bar{x}^{**} - \hat{x}|| \leq \sqrt{\varepsilon}$; and

(2) $||\bar{x}^* - x^*|| \leq \sqrt{\varepsilon}$.

Since A is of type (D) there exists a net $\{(x_\alpha, x_\alpha^*)\} \subset \mathrm{Gr}(A)$ such that $\{\hat{x}_\alpha\}$ is norm-bounded and converges to $\bar{x}^{**}$ in the weak star topology of X^{**}, while $\{x_\alpha^*\}$ norm-converges to $\bar{x}^*$ in X^*. Let α be so large that:

(a) $|\langle x_\alpha - x, x_\alpha^* - \bar{x}^*\rangle| \leq \varepsilon/2$;

(b) $|\langle \hat{x}_\alpha - \bar{x}^{**}, \bar{x}^* - x^*\rangle| \leq \varepsilon/2$;

(c) $||x^*_\alpha - \bar{x}^*|| \leq \sqrt{\varepsilon}$.

Condition (b) is a consequence of the weak star convergence of $\{\hat{x}_\alpha\}$ to $\bar{x}^{**}$, condition (c)–from the norm-convergence of $\{x^*_\alpha\}$ to $\bar{x}^*$, while condition (a) follows by the norm-convergence of $\{x^*_\alpha\}$ to $\bar{x}^*$ and the fact that $\{\hat{x}_\alpha\}$ is norm-bounded. Fix now some α as above and put $x_\varepsilon = x_\alpha$ and $x^*_\varepsilon = x^*_\alpha$. The condition (i) of the corollary is a direct consequence of (2) and (c) above. As to (ii) using (1),(2), (a) and (b) we have:

$$
\begin{aligned}
|\langle x_\varepsilon - x, x^*_\varepsilon - x^*\rangle| &= |\langle x_\alpha - x, x^*_\alpha - x^*\rangle| \\
&\leq |\langle x_\alpha - x, x^*_\alpha - \bar{x}^*\rangle| + |\langle x_\alpha - x, \bar{x}^* - x^*\rangle| \\
&\leq \frac{\varepsilon}{2} + |\langle \hat{x}_\alpha - \bar{x}^{**}, \bar{x}^* - x^*\rangle| + |\langle \bar{x}^{**} - \hat{x}, \bar{x}^* - x^*\rangle| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + ||\bar{x}^{**} - \hat{x}||\,||\bar{x}^* - x^*|| \\
&\leq \varepsilon + \sqrt{\varepsilon}\sqrt{\varepsilon} = 2\varepsilon.
\end{aligned}
$$

The proof is complete. ∎

4 Extended sum of monotone operators

Given two monotone operators $A, B : X \to X^*$, let us define the operator $A+B : X \to X^*$ as the usual pointwise sum of A and B:

$$(A + B)x = Ax + Bx, \quad x \in X.$$

This sum is always a monotone operator with domain $\mathrm{Dom}(A + B) = \mathrm{Dom}A \cap \mathrm{Dom}B$. But if A and B are maximal monotone operators, it does not follow, in general, that their sum $A + B$ is a maximal monotone operator. Some counterexamples are given in [Ph1] p. 54. Starting with the classical results of Browder [Br] and Rockafellar [Ro2], there have been a number of papers giving sufficient conditions when the sum of two maximal monotone operators is again a maximal monotone operator (see e.g. [A, ART, Ch, Si] just to mention a few).

As we mentioned in the introduction, the lack of maximal monotonicity for $A + B$ was a reason for different authors to look for a generalized notion of sum of monotone operators. Our aim here is to introduce a type of sum of A, B based on their enlargements. We define an *extended sum* of two monotone operators $A, B : X \to X^*$ at the point $x \in X$ by the formula

$$A \underset{ext}{+} B(x) = \bigcap_{\varepsilon>0} \overline{A^\varepsilon x + B^\varepsilon x}^{w^*}.$$

Obviously, $A + B \subset \overline{A + B} \subset A \underset{ext}{+} B$ and hence, $\mathrm{Dom}(A) \cap \mathrm{Dom}(B) \subset \mathrm{Dom}(A \underset{ext}{+} B)$. Observe also that the sum is commutative: $A \underset{ext}{+} B = B \underset{ext}{+} A$. We see below that in several important cases the extended sum is a maximal monotone operator.

Theorem 4.1 *Let $A, B : X \to X^*$ be two maximal monotone operators such that $\overline{A+B}$ is a maximal monotone operator. Then for any $x \in X$*

$$\overline{A+B}(x) = A \underset{ext}{+} B(x)$$

Proof: The left hand side is always included in the right hand side because of the definition. So we have to prove only the inverse inclusion. In order to show this we easily see first that for any $x \in X$ and any $\varepsilon > 0$ we have:

$$A^\varepsilon x + B^\varepsilon x \subset (A+B)^{2\varepsilon} x. \tag{4.1}$$

Indeed, fix some $x \in X$ and $\varepsilon > 0$ and let $x^* \in A^\varepsilon x + B^\varepsilon x$. Then $x^* = u^* + v^*$ for some $u^* \in A^\varepsilon x$ and $v^* \in B^\varepsilon x$. Take now an arbitrary couple (z, z^*) from $\mathrm{Gr}(A+B)$. Hence $z^* \in (A+B)(z)$ and consequently there are $z_1^* \in Az$ and $z_2^* \in Bz$ so that $z^* = z_1^* + z_2^*$. Now we have

$$\begin{aligned} \langle x - z, x^* - z^* \rangle &= \langle x - z, u^* + v^* - z_1^* - z_2^* \rangle \\ &= \langle x - z, u^* - z_1^* \rangle + \langle x - z, v^* - z_2^* \rangle \\ &\geq -\varepsilon - \varepsilon = -2\varepsilon. \end{aligned}$$

The above chain of inequalities shows that $x^* \in (A+B)^{2\varepsilon}(x)$. In this way (4.1) is proved.

Now Proposition 3.3 shows that $(A+B)^{2\varepsilon} = \overline{A+B}^{2\varepsilon}$. Hence by (4.1) we get

$$A^\varepsilon x + B^\varepsilon x \subset \overline{A+B}^{2\varepsilon}(x),$$

which on its turn gives that

$$\overline{A^\varepsilon x + B^\varepsilon x}^{w^*} \subset \overline{A+B}^{2\varepsilon}(x).$$

Therefore,

$$A \underset{ext}{+} B(x) = \bigcap_{\varepsilon>0} \overline{A^\varepsilon x + B^\varepsilon x}^{w^*} \subset \bigcap_{\varepsilon>0} \overline{A+B}^{2\varepsilon}(x) = \overline{A+B}(x).$$

The last equality is due to Proposition 3.1. The proof is complete. ■

Corollary 4.2 *Let $A, B : X \to X^*$ be maximal monotone operators such that $A+B$ is a maximal monotone operator. Then for any $x \in X$*

$$(A+B)(x) = \bigcap_{\varepsilon>0} (A^\varepsilon x + B^\varepsilon x) = A \underset{ext}{+} B(x).$$

Proof: We will prove the first equality. The second is then a consequence of the previous theorem. To establish the first equality we have only to prove that its right hand side is included in the left hand side of the same equality. From (4.1) above we have

$$\bigcap_{\varepsilon>0} (A^\varepsilon x + B^\varepsilon x) \subset \bigcap_{\varepsilon>0} (A+B)^{2\varepsilon}(x) = (A+B)(x),$$

the last equality being true by virtue of Proposition 3.1. Whence the desired inclusion. The proof is complete. ■

From the above corollary we get the following kind of Hiriart-Urruty and Phelps formula for the subdifferential of the sum of two proper convex lower semicontinuous functions with the new type of enlargements, when the w^*-closures can be removed.

Corollary 4.3 *Let $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions, such that the sum of their subdifferentials $\partial f + \partial g$ is a maximal monotone operator (which amounts to saying that $\partial f + \partial g = \partial(f+g)$). Then for every $x \in X$*

$$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \Big[(\partial f)^{\varepsilon} x + (\partial g)^{\varepsilon} x\Big] = \partial f \underset{ext}{+} \partial g(x).$$

A sufficient condition for the maximality of the pointwise sum of the two subdifferentials is the so-called Robinson-Rockafellar condition, which requires the origin to be in the algebraic core of the difference of the domains of f and g (see e.g. [Th1], Lemma 5 and Corollary 6).

More general (and more important) than the above particular corollary, is that, without any additional assumptions, the subdifferential of the sum of two proper convex lower semicontinuous functions is equal to the extended sum of their subdifferentials. Before giving this result, we recall an interesting representation of the subdifferential of the sum of two proper convex lower semicontinuous functions (see Penot [Pe] and Thibault [Th2]): *Let X be a Banach space and $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be proper convex lower semicontinuous functions. Then $(y, y^*) \in \partial(f+g)$ if and only if there are two nets $\{(y_\alpha, y^*_\alpha)\} \subset \partial f$ and $\{(z_\alpha, z^*_\alpha)\} \subset \partial g$, such that $\{y_\alpha\}$ and $\{z_\alpha\}$ converge strongly to y, $y^* = w^* - \lim(y^*_\alpha + z^*_\alpha)$, $\langle y_\alpha - y, y^*_\alpha\rangle \to 0$ and $\langle z_\alpha - y, z^*_\alpha\rangle \to 0$.* In fact, one has in addition that $f(y_\alpha) \to f(y)$ and $g(z_\alpha) \to g(y)$. The result above is a refinement of a previous similar one due to Thibault [Th1], Theorem 3. For brevity, here (and below) we use the convenience $(x, x^*) \in \partial f$ instead of $(x, x^*) \in \mathrm{Gr}(\partial f)$

Now we are ready to prove that the subdifferential of the sum of two convex proper lower semicontinuous functions is exactly the extended sum of their subdifferentials. In this way we have a kind of Hiriart-Urruty and Phelps formula with the new enlargements. This result, together with the above Theorem 4.1 and Corollary 4.2, show that in several important cases the extended sum is a maximal monotone operator. Observe that, in particular, the theorem below shows that the subdifferential of the sum and the extended sum of the subdifferentials have the same domain.

Theorem 4.4 *Let X be a Banach space and $f, g : X \to \mathbf{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. *Then for every $x \in X$ one has:*

$$\partial(f+g)(x) = \partial f \underset{ext}{+} \partial g(x).$$

Proof: By the Hiriart-Urruty and Phelps result (Theorem 2.2) and the remarks after the definition of the ε-enlargements, it is clear that the left hand side of the above equation is included in the right hand side. Therefore, it remains only to prove the inverse inclusion.

Take some $x \in X$ and suppose that $x^* \in \partial f \underset{ext}{+} \partial g(x)$. We will show that the pair (x, x^*) is monotonically related to $\partial(f+g)$. In view of the maximal monotonicity of $\partial(f+g)$ this will imply $x^* \in \partial(f+g)(x)$. So, take some arbitrary couple $(y, y^*) \in \partial(f+g)$ and fix it. We will prove that $\langle x - y, x^* - y^* \rangle \geq 0$. To this end let $\delta > 0$ be an arbitrary fixed positive number. Choose $\varepsilon > 0$ in such a way that $\varepsilon \leq \delta/8$ and fix it as well. Since $x^* \in \overline{(\partial f)^\varepsilon x + (\partial g)^\varepsilon x}^{w^*}$ we can find $u_\varepsilon^* \in (\partial f)^\varepsilon x$ and $v_\varepsilon^* \in (\partial g)^\varepsilon x$ such that:

$$|\langle x - y, x^* - u_\varepsilon^* - v_\varepsilon^* \rangle| \leq \frac{\delta}{8}. \tag{4.2}$$

Fix these u_ε^* and v_ε^*. On the other hand, by the result of Thibault [Th2] formulated above, there are two nets $\{(y_\alpha, y_\alpha^*)\} \subset \partial f$ and $\{(z_\alpha, z_\alpha^*)\} \subset \partial g$, such that $\{y_\alpha\}$ and $\{z_\alpha\}$ converge strongly to y, $y^* = w^* - \lim(y_\alpha^* + z_\alpha^*)$, $\langle y_\alpha - y, y_\alpha^* \rangle \to 0$ and $\langle z_\alpha - y, z_\alpha^* \rangle \to 0$. Let α be so large that:

$$\begin{aligned}
&\text{(i) } ||y - y_\alpha||\,||u_\varepsilon^*|| \leq \frac{\delta}{8}, \quad ||y - z_\alpha||\,||v_\varepsilon^*|| \leq \frac{\delta}{8} \\
&\text{(ii) } |\langle x - y, y_\alpha^* + z_\alpha^* - y^* \rangle| \leq \frac{\delta}{8} \\
&\text{(iii) } |\langle y - y_\alpha, y_\alpha^* \rangle| \leq \frac{\delta}{8}, \quad |\langle y - z_\alpha, z_\alpha^* \rangle| \leq \frac{\delta}{8}
\end{aligned} \tag{4.3}$$

Now by (4.2) we have:

$$\begin{aligned}
\langle x - y, x^* - y^* \rangle &= \langle x - y, u_\varepsilon^* + v_\varepsilon^* - y^* \rangle + \langle x - y, x^* - u_\varepsilon^* - v_\varepsilon^* \rangle \\
&\geq \langle x - y, u_\varepsilon^* + v_\varepsilon^* - y^* \rangle - \frac{\delta}{8}.
\end{aligned}$$

Further, having in mind (4.3) (ii), we obtain that for large α:

$$\begin{aligned}
\langle x - y, x^* - y^* \rangle &\geq \langle x - y, u_\varepsilon^* + v_\varepsilon^* - y^* \rangle - \frac{\delta}{8} \\
&= \langle x - y, u_\varepsilon^* + v_\varepsilon^* - y_\alpha^* - z_\alpha^* \rangle + \langle x - y, y_\alpha^* + z_\alpha^* - y^* \rangle - \frac{\delta}{8} \\
&\geq \langle x - y, u_\varepsilon^* - y_\alpha^* \rangle + \langle x - y, v_\varepsilon^* - z_\alpha^* \rangle - \frac{2\delta}{8} \\
&= \langle x - y_\alpha, u_\varepsilon^* - y_\alpha^* \rangle + \langle x - z_\alpha, v_\varepsilon^* - z_\alpha^* \rangle \\
&\quad + \langle y_\alpha - y, u_\varepsilon^* - y_\alpha^* \rangle + \langle z_\alpha - y, v_\varepsilon^* - z_\alpha^* \rangle - \frac{2\delta}{8} \\
&= \langle x - y_\alpha, u_\varepsilon^* - y_\alpha^* \rangle + \langle x - z_\alpha, v_\varepsilon^* - z_\alpha^* \rangle \\
&\quad + \langle y_\alpha - y, u_\varepsilon^* \rangle + \langle y - y_\alpha, y_\alpha^* \rangle + \langle z_\alpha - y, v_\varepsilon^* \rangle + \langle y - z_\alpha, z_\alpha^* \rangle - \frac{2\delta}{8}.
\end{aligned}$$

The first two terms on the right hand side of the last equality are greater or equal of ε because of the fact that $(x, u^*_\varepsilon) \in (\partial f)^\varepsilon$, $(y_\alpha, y^*_\alpha) \in (\partial f)$, $(x, v^*_\varepsilon) \in (\partial g)^\varepsilon$ and $(z_\alpha, z^*_\alpha) \in (\partial g)$. Therefore, taking into account (4.3) (i) and (4.3) (iii) we can continue the above chain of inequalities as follows:

$$\begin{aligned}\langle x-y, x^*-y^*\rangle &\geq -2\varepsilon - ||y-y_\alpha||\,||u^*_\varepsilon|| - \frac{\delta}{8} - ||y-z_\alpha||\,||v^*_\varepsilon|| - \frac{\delta}{8} - \frac{2\delta}{8} \\ &\geq -2\varepsilon - \frac{\delta}{8} - \frac{\delta}{8} - \frac{4\delta}{8} = -2\varepsilon - \frac{6\delta}{8} \geq -\delta.\end{aligned}$$

In the last inequality we have used that $\varepsilon \leq \delta/8$. Consequently, we have proved that

$$\langle x-y, x^*-y^*\rangle \geq -\delta$$

and since δ was arbitrary, this entails that $\langle x-y, x^*-y^*\rangle \geq 0$. The proof is complete. ■

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