The Banach-Mazur Game: History and Recent Developments*

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Contents

1 Introduction

Let X and Y be (everywhere at least Hausdorff) topological spaces and $F : X \to Y$ be a set-valued (equivalently multivalued) mapping between them. In the subsequent sections we will be interested in conditions which assure the existence of a continuous selection of F which is defined on a big enough part of the domain space X.

More precisely, recall that a subset A of X is called *residual* in X if its complement in X is of the first Baire category in X, i.e. $X \setminus A$ can be represented as a countable union of sets whose closures in X are nowhere dense in X. The space X is said to be a *Baire space* if the intersection of every countable family of open and dense subsets of X is dense in X. Among the known examples of Baire spaces are complete metric spaces, (locally) compact topological spaces and Čech complete spaces. We will freely use also the fact that any open set of a Baiare space is again a Baire space. More generally, any dense G_δ-subspace of a Baire space is also a Baire one. Finally, if Y is a topological space which contains a dense Baire subspace is a Baire space as well. Therefore, if the space X is Baire and A is its residual subset, then A contains a dense G_δ-subset of X and is considered to be a big subset of X from topological point of view. In other words, the residual subsets in X contain most of the points of X. A property which is fulfilled at the points of a residual subset of some Baire space X is called a *generic property*.

Comming back to our main goal, it can be precisely formulated as follows: given a set-valued mapping $F : X \to Y$ acting from the Baire space X into Y, can we find conditions under which there exist a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f : X_1 \to Y$ so that $f(x) \in F(x)$ for every $x \in X_1$. The latter means that f is a *selection* of F on the set X_1. We will investigate also the cases when the selection f is not necessarily a single-valued mapping, as well as situations when the mapping F coincides with its selection.

Our approach to obtain the above results involves the well-known Banach-Mazur game. We will see that the existence of selections as above is closely related to the existence of (special type of) winning strategies for one of the players in this game. Therefore, in the next section h we introduce this game, then we formulate and prove the main results (Theorem 4.3 and Theorem 4.5). We then apply the selection theorems to get various results in different branches like geometry of Banach spaces, best approximation theory etc. Other applications of these theorems to optimization will be given in the Sections 10-11.

2 The Banach-Mazur Game

The Banach-Mazur game appeared for the first time in the famous Scottish Book. This book which has an exciting and interesting story (see [U,MO] for details and other problems) was created in the period 1935-1941 in the town of Lwów which at that time was in Poland. A group of mathematicians, working at the University of Lwów, consisted of people whose names later became well-known, like St. Banach, S. Mazur, S. Ulam, H. Steinhaus and others. This group frequently used to discuss informally mathematical problems in one of the nearby caffés–The Scotish Caffé House (from where the book took its name). Following an idea of St. Banach a large notebook was bought and hidden by the waiter. Each time an intersting problem (or its solution) appeared in the discussion the waiter was asked to bring out the notebook in order to write down the problem (or the solution) and then to return it at the secret place. Many of the problems were accompanied by prizes-from a bottle of beer to a bottle of wine. It was the time when Functional Analysis and related mathematical fields were born and the book contains a number of interesting initial problems which later received their solution. Fortunately, the book survived the Second World War and was published for the first time by S. Ulam in 1957 in Los Alamos, USA. Later it had another edition as well as there was a conference devoted to it ([U, Mau]).

In the Scottish Book under problem No.43 appeared the following problem posed by Mazur: There are two players (Mazur denoted them by A and B) and a non-empty subset E of the real numbers $\mathbb{R}$. The game is defined as follows: A selects first a non-empty interval d_1, then B selects a non-empty sub-interval d_2 of d_1. Further, A continues by selecting a non-empty sub-interval d_3 of d_2 and B responds by choosing a non-empty sub-interval d_4 and so on. A wins if the intersection of all intervals $d_1, d_2, \dots$ has a common point with E. Otherwise B wins. Mazur had observed the following: if the complement of E is of the first Baire category in some interval, then the player A has a rule with which it will win. If the set E itself is of the first Baire category in $\mathbb{R}$ then the player B has a winning rule in this game. The question posed by Mazur (with prize a bottle of wine!) was whether these two conditions are also necessary for the players A and B to win correspondingly. The answer was given on August 4, 1935 by St. Banach and was affirmative. But a proof by Banach never appeared. Later this game took the names of Banach and Mazur and it seems it is the first infinite positional game with perfect information.

A proof of this result was announced in 1956 by Mycelski et all [MySwZi], but the proof was not published. In 1957 the proof of a (much) more general result was given by J. Oxtoby in [Ox]. He considered the following general setting containing as a partial case the above game: there are given a topological space X, a subset $E \subset X$ of it and a family $\mathcal{W}$ of subsets of X with the following properties: (i) each $W \in \mathcal{W}$ contains a non-empty open subset of X; and (ii) each non-empty open subset of X contains an element of $\mathcal{W}$. Two players, Player I and Player II, choose alternatively elements from $\mathcal{W}$ $U_1 \supset U_2 \supset U_3 \supset \cdots$ (the choices of Player I are the sets with odd indexes and those of Player II, the sets with even indexes). Player I wins if the intersection $E \cap (\cap_{i=1}^{\infty} U_i)$ is not empty. Otherwise Player II wins. Oxtoby showed that: Player II has a winning rule in this game (see below the precise definitions of a winning strategy) if and only if the set E is of the first Baire category in X; moreover, if X is supposed to be a complete metric space then Player I has a winning rule in the game if and only if the set E has a complement which is of the first Baire category in some open subset of X.

Further generalizations and variants of this game (and discussion on many other topological games) as well as more details can be found in the excellent survey paper of Telgársky [Tel] (see also the book of Choquet [Ch2]). We will concentrate ourselves on the following, probably most known, modification of the Banach-Mazur game. In the above scheme we take $E = X$ and the family $\mathcal{W}$ consists of all non-empty open sets in X. Two players, who from now on we will designate by the commonly accepted notation α and β, play a game in the following way: β chooses first a non-empty open subset U_1 of X. Then α chooses a non-empty open set $V_1 \subset U_1$. Further, β chooses again a non-empty open subset $U_2 \subset V_1$ and again α selects a non-void open subset of the set chosen by β. The so obtained sequence of non-empty open sets $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ is called *a play.* The player α wins this play if $\cap_{i=1}^{\infty} U_i = \cap_{i=1}^{\infty} V_i \neq \emptyset$. Otherwise β wins. This game is usually denoted by $BM(X)$.

A *partial play* in the above game is any finite sequence of non-empty open sets of the type $U_1 \supset V_1 \supset \cdots \supset W$, where either $W = U_n$ or $W = V_{n-1}$ (or $W = V_n$), $n \geq 1$, representing the first n legal moves of the player β and the first $n-1$ (or n) moves of the player α.

A *strategy* s for the player α is called every rule (or a mapping) which assigns to each partial play $U_1 \supset V_1 \supset \cdots \supset V_{n-1} \supset U_n$, $n \geq 1$, a non-empty open set $V_n = s(U_1, V_1, \ldots, V_{n-1}, U_n) \subset U_n$. When α makes his/her choice with the help of the strategy s we call the resulting play an *s-play* (i.e. $V_i = s(U_1, V_1, \ldots, V_{i-1}, U_i) \subset U_i$ for every i). The strategy s is called *winning* for the player α (equivalently *α-winning*) if for every s-play $p = \{U_i, V_i\}_{i=1}^{\infty}$ the *target set* $T(p) = \cap_{i=1}^{\infty} U_i = \cap_{i=1}^{\infty} V_i$ is not empty. Similarly one defines a strategy (and winning strategy) for the player β in this game. Given a strategy s, by T_s we will denote the *target space* generated by this strategy, i.e. the set $T_s := \cup\{T(p) : p$ is an s-play in the game $BM(X)\}$.

Let us stress the fact that in the above definition of a strategy, the choice of the corresponding player at step n depends on all choices preceding this step. Of course of interest are also the strategies which depend only on the last move of the opponent: *a stationary strategy* (called also *tactics*) for the player α in the Banach-Mazur game $BM(X)$ is a mapping t from the family of all non-empty open sets of X into the same family with the property $t(U) \subset U$ for every non-empty open set $U \subset X$. The stationary strategy t for the player α is called winning (equivalently, *α-winning stationary strategy*, or *α-winning tactics*), if whenever one has a sequence of non-empty open sets $\{U_i\}_{i=1}^{\infty}$ with the property $U_{i+1} \subset t(U_i)$ for every i, it follows that $\cap_{i=1}^{\infty} U_i \neq \emptyset$. One defines analogously the notion of a stationary strategy (tactics) for the player β in $BM(X)$.

Of course every (winning) stationary strategy is also (winning) strategy for the same player. The converse is true if the player β is concerned, but not for the player α. Namely, (see Galvin and Telgársky [GaTel], Corollary $1'$): The player β has a stationary winning strategy in the game $B(X)$ if, and only if, it has a winning strategy in $BM(X)$. The same assertion for the player α is not true in general: G. Debs has constructed in [De] a completely regular topological space which admits an α-winning strategy in the game $BM(X)$ but which does not admit any α-winning stationary strategy in the same game.

Sometimes the topological spaces X which admit α-winning strategy in the game $BM(X)$ are called *weakly α-favorable*, and those which admit α-winning tactics–*α-favorable* (cf. [Tel, Wh]. An obvious example of a α-favorable space is any compact topological space X: given a non-empty open subset U of X the image of the tactis is defined to be any fixed open subset V of U which is included in U together with its closure. It is clear also that every complete metric space X is α-favorable: the definition of the α-winning tactics in this case is also easy: given a non-empty open set $U \subset X$, define $t(U)$ as a non-empty open ball so that the closure of $t(U)$ is a subset of U and its diameter is strictly less than one half the diameter of U. It is easily verified that the so defined t is a winning tactics for the player α in $BM(X)$. We will give Sections 10-11 characterizations of the weak α-favorability as well as characterizations of the α-favorability in the class of the metric spaces. Here we give a characterization of the absence of a winning strategy for the second player in $BM(X)$.

Theorem 2.1 *The topological space X is Baire space if and only if X does not admit a winning strategy for the player β in the game $BM(X)$.*

Proof: Let X be a Baire space and suppose that β possesses a winning strategy t in the game $BM(X)$. Let U_1 be the first choice of β according to this srategy. We will show that U_1 is of the first Baire category, which will be a contradiction. Indeed, put $U_0 = V_0 := X$ and let $\gamma_1 := \{(U_1, V_0)\}$. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a family of couples of open sets which is maximal with rtespect to the following properties:

(i) for every $n \geq 2$ and $(U_n, V_{n-1}) \in \gamma_n$ there is $(U_{n-1}, V_{n-2}) \in \gamma_{n-1}$ with $V_{n-1} \subset U_{n-1}$;

(ii) for every n the family $\{U : (U, V) \in \gamma_n \text{ for some } V\}$ is paire-wise disjoint;

(iii) for every $n \geq 2$ and $(U_n, V_{n-1}) \in \gamma_n$ there exists a (by (ii), uniquely determiened) sequence of couples $(U_i, V_{i-1}) \in \gamma_i$, $i = 1, \ldots, n-1$, so that $(U_1, V_1, ..., U_{n-1}, V_{n-1})$ is a partial play in the game $BM(X)$ and, moreover, $U_i := t(U_1, \ldots, V_{i-1})$ for every $i = 2, \ldots, n$.

Put $W_n := \cup\{U_n : (U_n, V_{n-1}) \in \gamma_n \text{ for some } V_{n-1}\}$, $n = 1, 2, \ldots$. We will show that each W_n is dense in U_1. For $n = 1$ this is clear. Let the assertion is true up to some $n \geq 1$ and suppose that W_{n+1} is not dense in U_1. Then there is some non-empty open set $U_0 \subset U_1$ so that $U_0 \cap W_{n+1} = \emptyset$. On the other hand, $W_n \cap U_1 \neq \emptyset$ and hence, there is some couple $(U_n, V_{n-1}) \in \gamma_n$ with $U_n \cap U_0 \neq \emptyset$. By (iii) above, there is a sequence $(U_i, V_{i-1}) \in \gamma_i$ for $i = 1, \ldots, n-1$, so that $(U_1, V_1, \ldots, U_{n-1}, V_{n-1})$ is a partial play in the Banach-Mazur game and $U_n = t(U_1, V_1, \ldots, U_{n-1}, V_{n-1})$. Set $V_n := U_n \cap U_0$ and let $U_{n+1} := t(U_1, V_1, \ldots, U_n, V_n)$. Then the sequence of families $\{\gamma_i'\}$ where $\gamma_i' := \gamma_i$ for $i \neq n+1$ and $\gamma_{n+1}' := \gamma_{n+1} \cup \{(U_{n+1}, V_n)\}$ is stritcly larger than $\{\gamma_i\}$ and still satisfies (i)-(iii). The achieved contradiction shows that each W_n is dense in U_1.

Now, we claim that $\cap_{n=1}^{\infty} W_n = \emptyset$. Assume the contrary and take some $x \in W_n$ for every $n = 1, 2 \ldots$. By (i)-(iii) above there is a (uniquely determined) sequence $(U_n, V_{n-1}) \in \gamma_n$, $n \geq 1$, so that $x \in U_n$ for each n and the sequence $\{(U_n, V_n)\}$ is a play in the game $BM(X)$ in which every choice of the player β is obtained by using the strategy t, i.e. $U_n = t(U_1, V_1, \dots, U_{n-1}, V_{n-1})$ for every $n \geq 1$. Then $\cap_{n=1}^{\infty} U_n = \emptyset$ contradiction the choice of x. Hence, $\cap_{n=1}^{\infty} W_n = \emptyset$ and consequently (remember that every W_n was dense in U_1) the set U_1 is of the first Baire category. The proof of this implication is completed.

Conversely, let the space X do not admit a winning strategy for the player β in the game $BM(X)$. Suppose that the space X is not Baire. This means that there is a non-empty open set U_1 which is of the first Baire category in X. I.e. there is a sequence $\{A_n\}$ of closed nowhere dense sets in X so that $\cup_{n=1}^{\infty} A_n \supset U_1$. Define the following (stationary) strategy for the player β: for $n = 1$ the choice of β is U_1; for $n \geq 2$, given a non-empty open $V_n \subset U_1$ we put $t(V_n) := V_n \setminus A_n$. It is easily seen that the so constructed t is a winning (stationary) strategy for the player β in the game $BM(X)$. This is a contradiction. Hence, the space X is a Baire space. The proof of the theorem is completed. ■

Topological spaces X which do not admit β-winning strategy in the game $BM(X)$ are called *β-defavorable*.

3 Preliminary facts about set-valued mappings

Let F be a set-valued mapping acting between the topological spaces X and Y. First we introduce a piece of notation. Since we will consider mappings which may have also empty images, the symbol $\mathrm{Dom}\,(F)$ will denote, as usual, the *domain* of F, i.e. the set

$$\mathrm{Dom}\,(F) := \{x \in X \colon F(x) \neq \emptyset\}.$$

For $A \subset X$ its immage under F is the set $F(A) := \cup\{F(x) \colon x \in A\}$, and for $B \subset Y$ the two possible preimages of B under F are:

$$F^{-1}(B) := \{x \in X \colon F(x) \bigcap B \neq \emptyset\}$$

and

$$F^{\#}(B) := \{x \in X \colon F(x) \subset B\}.$$

Observe that $F^{\#}(B)$ contains each point $x \in X$ with $F(x) = \emptyset$. Let us mention also that $F^{-1}(Y) = \mathrm{Dom}\,(F)$ and $F^{\#}(Y) = X$. Finally, the set

$$\mathrm{Gr}\,(F) := \{(x, y) \in X \times Y : y \in F(x)\},$$

is the *graph* of F,

Further, we recall some continuity-like properties of set-valued mappings. The mapping $F : X \to Y$ is called *upper* (resp. *lower*) semicontinuous at a point $x_0 \in X$ if for every open $V \subset Y$ with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) there is an open set

$U \subset X$ with $x_0 \in U$ such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$) whenever $x \in U$. We abbreviate this by F is usc (resp. lsc) at x_0. F is usc (resp. lsc) in X if it is usc (resp. lsc) at any point of X. Equivalently, F is usc (resp. lsc) in X if for every open $V \subset Y$ the set $F^{\#}(V)$ (resp. the set $F^{-1}(V)$) is open in X.

In what follows, most of our examples will involve mappings F with domain $\mathrm{Dom}\,(F)$ which is **dense** in the domain space X. The reason is obvious: if x_0 is a point outside the closure (in X) of $\mathrm{Dom}\,(F)$ then for some open set U of X containing x_0 we have $F(x) = \emptyset$ for every $x \in U$. I.e. F is usc and lsc at each such point. When we say the F is non-empty valued we mean that $\mathrm{Dom}\,(F) = X$. Let us observe also that, if $\mathrm{Dom}\,(F)$ is dense in X and if F is usc at some $x \in X$, then $F(x) \neq \emptyset$.

The mapping F is called *usco* in X (or at a point $x_0 \in X$) if it is usc and compact-valued in X (or at x_0). For every usco mapping $F : X \to Y$ its grap $\mathrm{Gr}\,(F)$ is a closed subset of $X \times Y$ considered with the product topology. When $\mathrm{Gr}\,(F)$ is closed in $X \times Y$ we say that F has a *closed graph.* Sometimes the converse is also true: for instance, if $F : X \to Y$ has a closed graph and Y is compact then F is usco.

An usco $F : X \to Y$ is *minimal* if its graph does not contain properly the graph of any other usco $G : X \to Y$ with the same domain. Let us mention that every non-empty valued mapping $G : X \to Y$ with closed graph which is contained in an usco mapping $F : X \to Y$ (that is $G(x) \subset F(x)$ for every $x \in X$) is usco itself. Hence, by Kuratowski-Zorn lemma every usco mapping $F : X \to Y$ contains a minimal usco $G : X \to Y$.

We will go further by considering minimal elements in a larger class of mappings. Below, as usual, for a subset A of a topological space X, we designate by $\mathrm{Int}_X(A)$ and $\mathrm{Cl}_X(A)$ the interior and the closure of the set A in X. If there is no danger of confusion, we will omit the subscript X.

First, we give a relaxation of the notion of semicontinuity: a mapping $F : X \to Y$ is called *upper (resp. lower) quasicontinuous* at $x_0 \in X$ if for every open set V of Y with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) there is an open set $U \subset X$ with $x_0 \in \mathrm{Cl}\,(U)$ such that $F(x) \subset V$ (resp. $F(x) \cap \neq \emptyset$) whenever $x \in U$.

Given topological spaces X and Y we will consider the class $QC(X, Y)$ of set-valued mappings $F : X \to Y$ which are upper quasicontinuous at any $x \in \mathrm{Dom}\,(F)$ and, moreover, have closed graph. It is easily seen that, F is upper quasicontinuous at any $x \in \mathrm{Dom}\,(F)$ if and only if for every open V in Y the set $\mathrm{Int}\, F^{\#}(V)$ is dense in $F^{\#}(V) \cap \mathrm{Dom}\,(F)$. Every usco $F : X \to Y$ (even every closed-valued usc $F : X \to Y$, with Y regular) is in the above class. Evidently, every mapping F from $QC(X, Y)$ with Y compact, is usco in X.

We call, as above, a closed graph mapping $F : X \to Y$ which is upper quasicontinuous at any point of $\mathrm{Dom}\,(F)$, minimal if its graph is a minimal element, with respect to the set inclusion order in $X \times Y$, in the family of all graphs of mappings

G from $QC(X,Y)$ with $\mathrm{Dom}\,(G) = \mathrm{Dom}\,(F)$.

Here we cannot claim that every $F \in QC(X,Y)$ contains a minimal one (of the same type) with the same domain because of lack of compactness of the images. But the minimal mappings from $QC(X,Y)$ still share the basic properties of the minimal usco ones. The next proposition is well-known for minimal usco mappings.

Proposition 3.1 *The following are equivalent for a closed graph mapping $F : X \to Y$ which is upper quasicontinuous at any point of* $\mathrm{Dom}\,(F)$*:*

(a) F is minimal;

(b) for every open U in X and closed B in Y from $F(x) \cap B \neq \emptyset$ for every $x \in U \cap \mathrm{Dom}\,(F)$, it follows that $F(U) \subset B$;

(c) if U and V are open subsets of X and Y such that $U \cap F^{-1}(V) \neq \emptyset$ then there is a non-empty open $U' \subset U$ with $F(U') \subset V$.

Proof: First of all observe that, without loss of generality,we may assume that $\mathrm{Dom}\,(F)$ is dense in X.

To prove (a)⇒(b), let $U \subset X$ be open and $B \subset Y$ be closed in Y such $F(x) \cap B \neq \emptyset$ for every $x \in U \cap \mathrm{Dom}\,(F)$. We show that $F(U) \subset B$. Suppose that the set $A := \{x \in U : F(x) \setminus B \neq \emptyset\}$ is not empty and define $G : X \to Y$ by

$$G(x) = \begin{cases} F(x) & \text{if } x \in X \setminus U \\ F(x) \bigcap B & \text{if } x \in U. \end{cases}$$

By the assumptions $\mathrm{Dom}\,(G) = \mathrm{Dom}\,(F)$. Moreover, since A is non-empty the graph of F is strictly larger than the graph of G. Therefore, to get a contradiction, it remains to show that G is with closed graph and upper quasicontinuous at any point of $\mathrm{Dom}\,(F)$.

First, it is easily verified that $\mathrm{Gr}\,(G) = \mathrm{Gr}\,(F) \setminus (U \times (Y \setminus B))$. Hence G has a closed graph.

Further, let $x_0 \in \mathrm{Dom}\,(G) = \mathrm{Dom}\,(F)$. If $x_0 \in U$ then $G(x_0) = F(x_0)$ and the upper quasicontinuity of G at x_0 follows by the upper quasicontinuty of F at x_0. So suppose $x_0 \notin U$ and let $G(x_0) \subset V$ with V open in Y. Then $F(x_0) \subset V' := V \cup (Y \setminus B)$ and since $Y \setminus B$ is open, there is some open subset W of X with $x_0 \in \mathrm{Cl}\,(W)$ and $F(x) \subset V'$ for each $x \in W$. We may think that $W \subset U$ and then it is easily seen that $G(x) \subset V$ for any $x \in W$. Therefore, G is upper quasicontinuous at each $x \in \mathrm{Dom}\,(G)$.

(b)⇒(c). Let $F : X \to Y$ be a mapping from the class $QC(X,Y)$ satisfying (b), and U and V be open subsets of X and Y correspondingly such that $U \cap F^{-1}(V) \neq \emptyset$. Then, because of (b), there is at least one point $x_0 \in U \cap \mathrm{Dom}\,(F)$ with $F(x_0) \subset V$. Since F is upper quasicontinuous at x_0 we have some nonempty open set U' of X (we may think $U' \subset U$) so that $F(U') \subset V$. This completes the proof of the implication.

(c)$\Rightarrow$(a). Let F satisfy (c). Suppose there are a mapping $G \in QC(X,Y)$ with $\mathrm{Dom}\,(G) = \mathrm{Dom}\,(F)$ which is contained in F and $x_0 \in \mathrm{Dom}\,(F)$ such that $F(x_0) \setminus G(x_0) \neq \emptyset$. Take $y_0 \in F(x_0) \setminus G(x_0)$. This means $(x_0, y_0) \notin \mathrm{Gr}\,(G)$. Since $\mathrm{Gr}\,(G)$ is closed there are open sets U in X and V in Y with $x_0 \in U$ and $y_0 \in V$ such that $(U \times V) \cap \mathrm{Gr}\,(G) = \emptyset$. On the other hand, $(x_0, y_0) \in \mathrm{Gr}\,(F)$ giving $U \cap F^{-1}(V) \neq \emptyset$. By c) there is a non-empty open set $U' \subset U$ with $F(U') \subset V$. Since $\mathrm{Dom}\,(F)$ is dense in X we get $x \in U' \cap \mathrm{Dom}\,(F)$ with $F(x) \subset V$. But $\mathrm{Dom}\,(F) = \mathrm{Dom}\,(G)$, hence $\emptyset \neq G(x) \subset V$. This is a contradiction. The proof of this implication and the proposition is completed. ■

We mentioned above that each mapping $G : X \to Y$ which has a closed graph and is contained in an usco mapping $F : X \to Y$ is usco itself. Hence, the following is immediate.

Proposition 3.2 *Every minimal usco mapping $F : X \to Y$ is minimal also in the class $QC(X,Y)$ of closed graph mappings between X and Y which are upper quasicontinuous at any point of their domain.*

We give some simple examples showing that the class of (minimal) mappings from $QC(X,Y)$ is strictly larger than the class of (minimal) usco mappings. Recall that a mapping $F : X \to Y$ is called *open* if for every non-empty open set $U \subset X$ the set $F(U)$ is (non-empty) and open in Y. Analogously, F is called *closed* if for every closed set $B \subset X$ the set $F(B)$ is closed in Y.

Example 3.3 Let Y be a dense subspace of the space X. Consider the mapping $F : X \to Y$ defined by $F(x) := x$ provided $x \in Y$ and $F(x) := \emptyset$ if $x \in X \setminus Y$. Obviously $\mathrm{Dom}\,(F) = Y$. It is easily seen that the mapping F is from the class $QC(X,Y)$ which is open, closed and minimal. F is usco iff $X = Y$. F is lsc iff the space Y is open in X.

Example 3.4 Let us consider a single-valued mapping $f : Y \to X$ from the topological space Y into the topological space X such that the set $Z = f(X)$ is dense in X. Put $F(x) := f^{-1}(x)$ if $x \in Z$ and $F(x) := \emptyset$ if $x \in X \setminus Z$. Suppose f is continuous and closed with respect to the induced topology on $f(X)$. Then the mapping F is open, closed and belongs to the class $QC(X,Y)$. Moreover, if in addition f is *irreducible* (that means $f(B) \neq f(Y)$ whenever $B \subset Y$ is closed and $B \neq Y$) then F is minimal. The mapping F is usco iff $Z = X$ and f is closed. Finally, F is lsc iff f is open and Z is open in X.

Example 3.5 Let $G : Z \to Y$ be a non-empty valued mapping and Z be dense in some topological space X. Define a mapping $F : X \to Y$ in a natural way: $F(x) := G(x)$ provided $x \in Z$ and $F(x) := \emptyset$ otherwise. The mapping G is in the class $QC(Z,Y)$ exactly when the mapping F is in the class $QC(X,Y)$. G is minimal in $QC(Z,Y)$ iff F is minimal in $QC(X,Y)$.

4 Main selection theorems

In this section we formulate and then prove our main selection theorems. To start we need two notions. The first is a relation between a given set-valued mapping and a strategy in the range space.

Definition 4.1 *Let X, Y be topological spaces, $F : X \to Y$ be a set-valued mapping, s be a strategy for the player α in the game $BM(Y)$. We say that F and s are coordinated (resp. strictly coordinated) if for every partial s-play $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n$ (i.e. $V_i = s(U_1, V_1, \ldots, U_i)$ for every $i = 1, \ldots, n$) the set $\cup\{\mathrm{Int}\,\mathrm{Cl}\, F^{-1}(V) : V = s(U_1, V_1, \ldots, U_n, V_n, U)$ for some non-empty open $U \subset V_n\}$ is dense in the set $\mathrm{Int}\,\mathrm{Cl}\, F^{-1}V_n$ (resp. $\cup\{\mathrm{Int}\, F^{-1}(V) : V = s(U_1, V_1, \ldots, U_n, V_n, U)$ for some non-empty open $U \subset V_n\}$ covers the set $\mathrm{Int}\, F^{-1}(V_n)$).*

The second notion concerns special type of winning strategies for the player α in the game $BM(Y)$. Let us remind that a nested sequence of open sets $\{U_n\}_{n=1}^{\infty}$ of a topological space Y is called *complete* if the intersection $\cap_{n=1}^{\infty} U_n$ is non-empty and compact and the family $\{U_n\}_{i=n}^{\infty}$ is a base of neighborhoods for this intersection, i.e. for every open $U \supset \cap_{n=1}^{\infty} U_n$ there exists some m with $U_m \subset U$.

Definition 4.2 *An α-winning strategy s in the game $BM(Y)$ is said to be complete if for every s-play $p = \{U_i, V_i\}_{i=1}^{\infty}$ the sequence $\{U_n\}_{n=1}^{\infty}$ (as well as the sequence $\{V_n\}_{n=1}^{\infty}$) is complete.*

In the next section we will give rather general sufficient conditions for given F and s to be coordinated, as well as for the existence of complete winning strategies. Here we continue by intruducing two mappings which are extensions to a set-valued mapping $F : X \to Y$. Namely, given such an F, let $F^* : X \to Y$ and $\bar{F} : X \to Y$ be difend as follows:

$$F^*(x) = \bigcap\{\mathrm{Cl}\, F(W) : W \text{ open in } X,\, x \in W\}, \quad x \in X,$$

and

$$\bar{F}(x) = \mathrm{Cl}\, F(x), \quad x \in X.$$

It is well-known that the mapping F^* has a closed graph, and that the graph of F is closed if, and only if, $F(x) = F^*(x)$ for every $x \in X$. On the other hand, it is evident that $\mathrm{Dom}\,(F) = \mathrm{Dom}\,(\bar{F})$ and that $F(x) = \bar{F}(x)$ for each $x \in X$ if and only the mapping F has closed images.

We are ready to give our first selection theorem:

Theorem 4.3 *Let X be a Baire space and Y be a topological space with a complete α-winning strategy s and a target space T_s. Suppose $F : X \to Y$ is a set-valued* *map so that* Dom (F) *is dense in* X *and* F *is coordinated with* s*. Then there exist a dense* G_δ*-subset* $X_1 \subset X$ *and a non-empty valued usco map* $G : X_1 \to T_s \subset Y$ *which is a selection of* F^* *(i.e. for every* $x \in X_1$ *we have* $G(x) \subset F^*(x)$*). In particular, if* F *itself has a closed graph then* $X_1 \subset$ Dom (F) *and* G *is am usco selection of* F *on* X_1*.*

If the complete strategy s *is such that for every* s*-play* $p = \{U_i, V_i\}_{i=1}^\infty$ *the target set* $T(p) = \cap_{i=1}^\infty U_i$ *is a singleton, then the mapping* G *is single-valued and continuous at each point of* X_1*. Moreover, if* F *is non-empty valued and* F *and* s *are strrictly coordinated then* G *is a single-valued selection also of* $\bar{F}$*.*

Before proving this theorem we will formulate our second selection theorem. It is concerned with the case when the given mapping F coincide with its selection. For this we need another relation between the mapping and the complete strategy.

Definition 4.4 *We say that the mapping* $F : X \to Y$ *and the strategy* s *in* Y *are strongly coordinated if for every partial* s*-play* $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n$ *the set* $\cup\{\mathrm{Int}\, F^\# V : V = s(U_1, V_1, \ldots, U_n, V_n, U)$ *for some non-empty open* $U \subset V_n\}$ *is dense in the set* $\mathrm{Int}\, F^\# V_n$*.*

Now we have our second selection theorem:

Theorem 4.5 *Let* X *be a Baire space,* Y *be a regular topological space with a complete* α*-winning strategy* s *and a target space* T_s*. Let* $F : X \to Y$ *be a set-valued map such that* Dom (F) *is dense in* X *and* F *is strongly coordinated with* s*. Then there exists a dense* G_δ*-subset* $X_1 \subset X$ *such that the restriction of* F^* *on* X_1 *is an usco mapping from* X_1 *into* T_s*. In particular, if* F *has a closed graph, then the restriction* $F|X_1$ *is an usco mapping from* X_1 *into* T_s*.*

If the complete strategy s *is such that for every* s*-play* $p = \{U_i, V_i\}_{i=1}^\infty$ *the target set* $T(p) = \cap_{i=1}^\infty U_i$ *is a singleton, then the mapping* F^* *(and hence, in case* F *has a closed graph, the mapping* F*) is single-valued and upper semicontinuous at each point of* X_1*. Moreover, in this last case if in addition* F *is non-empty valued then* $\bar{F}$ *(and hence, in case* F *has closed images, the mapping* F*) is single-valued and upper semicontinuous at each point of* X_1*.*

Let us prove now our selection theorems. We will first establish a lemma which is a key point in the proofs. In the sequel, given a set-valued map $F : X \to Y$, under an *admissible pair* (resp. *strictly admissible*) we mean a couple (W, V) of non-empty open subsets of X and Y, respectively, such that $F^{-1}V$ is dense in W (resp. $W \subset F^{-1}(V)$).

Lemma 4.6 *In the assumptions of Theorem 4.3, let* $(U_1, V_1, \ldots U_n, V_n)$ *be a partial* s*-play and* W_n *be a non-empty open subset of* X *such that* (W_n, V_n) *is an admissible* *pair. Suppose further that $\Gamma(W_n) = \{(W, U, V)\}$ is a maximal system of triples such that:*

(i) U is a non-empty open subset of V_n;

(ii) $V = s(U_1, V_1, \ldots, U_n, V_n, U)$ (i.e. (U, V) is a continuation of the partial play $(U_1, V_1, \ldots, U_n, V_n)$;

(iii) (W, V) is an admissible pair, so that $W \subset W_n$;

(iv) $\gamma(W_n) := \{W : (W, U, V) \in \Gamma(W_n)\}$ is a pairwise disjoint system of open subsets of W_n.

Then the set $H(W_n) := \cup\{W : W \in \gamma(W_n)\}$ is dense in W_n.

If, F is non-empty valued and F and s are strictly coordinated, then all admissible couples above can be taken strictly admissble.

Proof: Suppose the contrary and take some non-empty open $W' \subset W_n \setminus \mathrm{Cl}\, H(W_n)$. Since F and s are coordinated and (W_n, V_n) is an admissible pair (i.e. $W_n \subset \mathrm{Cl}\, F^{-1}(V_n)$) there exists some continuation (U_{n+1}, V_{n+1}) of the partial s-play $(U_1, V_1, \ldots, U_n, V_n)$ such that $W := W' \cap \mathrm{Int}\, \mathrm{Cl}\, F^{-1}(V_{n+1}) \neq \emptyset$. Evidently, (W, V_{n+1}) is an admissible pair for which the triple (W, U_{n+1}, V_{n+1}) does not belong to $\Gamma(W_n)$. On the other hand, $\Gamma(W_n)$ together with the latter triple is a system which satisfies (i)-(iv). This contradicts the maximality of the system $\Gamma(W_n)$. Observe that if the couple (W_n, U_n) above was strictly admissible and F and s are strictly coordinated, with F non-empty valued, then the couple (W, V_{n+1}) would be strictly admissble as well. The proof of the lemma is completed. Let us observe that if the couple (W_n, U_n) was strictly admissible and F and s are strictly coordinated then the couple (W, V_{n+1}) would be strictly admissble as well. Let us note that the couple (X, Y) is admissible because $\mathrm{Dom}\,(F)$ is dense in X (and strictly admissble if $\mathrm{Dom}\,(F) = X$). Therefore, the lemma can be proved also for $n = 0$ provided we put $W_0 = X$, $U_0 = V_0 = Y$. ■

Proof of Theorem 4.3: Let $W_0 = X$, $U_0 = V_0 = Y$ and set $\gamma_0 := \{X\}$. Apply the lemma for $n = 0$. As a result we get the maximal system of triples $\Gamma(W_0)$ satisfying (i)-(iv). Put $\Gamma_1 := \Gamma_(W_0)$, $\gamma_1 := \gamma(W_0)$ and $H_1 := H(W_0)$. Observe that γ_1 is a disjoint system of open sets in X whose union H_1 is dense in X. Therefore, for every $W_1 \in \gamma_1$ there exists a uniquely determined triple $(W_1, U_1, V_1) \in \Gamma_1$. Applying the lemma for this triple we get a system of triples $\Gamma(W_1)$ satisfying (i)-(iv). Put $\Gamma_2 := \cup\{\Gamma(W_1) : W_1 \in \gamma_1\}$, $\gamma_2 := \cup\{\gamma(W_1) : W_1 \in \gamma_1\}$ and $H_2 := \cup\{H(W_1) : W_1 \in \gamma_1\}$. As above, one sees that γ_2 is a disjoint system of open sets in X which is inscribed in γ_1 and whose union is (open and) dense in X.

Continuing in this manner we construct a sequence of families $\{\Gamma_n\}_{n=1}^{\infty}$ of triples and a sequence of disjoint families $\{\gamma_n\}_{n \geq 0}$ of open sets in X, with $\gamma_0 = \{X\}$, such that for every $n \geq 1$ we have:

(a) Γ_n is a union of the families $\Gamma(W_{n-1})$, $W_{n-1} \in \gamma_{n-1}$, where $\Gamma(W_{n-1})$ is obtained by lemma 4.6 from some uniquely determined partial play $(U_1, V_1, \ldots, U_{n-1}, V_{n-1})$;

(b) γ_n is a union of the families $\gamma(W_{n-1})$ from the condition (iv) of the Lemma;

(c) the set $H_n\colon = \cup\{W_n\colon W_n \in \gamma_n\}$ is open and dense in X.

Put $X_1 := \cap_{n=1}^{\infty} H_n$. This is a dense and G_δ-subset of X. Every $x \in X_1$ uniquely determines a sequence of triples $(W_n(x), U_n(x), V_n(x))_{n=1}^{\infty}$ such that $x_1 \in \cap_{n=1}^{\infty} W_n(x)$, $(W_n(x), V_n(x))$ is an admissible pair for every n and the sequence $\{U_n(x), V_n(x)\}_{n=1}^{\infty}$ is an s-play. Observe that, if F ans s are strictly coordinated, then according to Lemma 4.6 the sequence $\{W_n(x), V_n(x)\}$ consists of strictly admissble pairs.

Since the strategy s is a complete α-winning strategy the set $\Phi(x) := \cap_{n=1}^{\infty} U_n(x) = \cap_{n=1}^{\infty} V_n(x)$ is non-empty and compact. The so defined set-valued map $\Phi : X_1 \to T_s$ is also usc. Indeed, take some open $U \supset \Phi(x_0)$, where $x_0 \in X_1$. Since s is complete the sequence $\{U_n(x_0)\}_{n=1}^{\infty}$ is a base for the compact set $\Phi(x_0)$. Hence, there exists some n with $\Phi(x_0) \subset U_n(x_0) \subset U$. Now, if $x \in W_n(x_0) \cap X_1$, then we have (because of (iv) of the lemma) that $U(x) = U(x_0)$ and hence $\Phi(x) \subset U_n(x_0) \subset U$. I.e. Φ is usco.

We prove further that $\Phi(x) \cap F^*(x) \neq \emptyset$ for every $x \in X_1$ (thus we show also that $\mathrm{Dom}(F^*) \supset X_1$). Suppose that for some $x_0 \in X_1$ we have $\Phi(x_0) \cap F^*(x_0) = \emptyset$. Since the graph of F^* is closed in $X \times Y$ and $(x_0, \Phi(x_0)) \cap \mathrm{Gr}\,(F^*) = \emptyset$ we can find some open $U \supset \Phi(x_0)$ and an open W containing x_0 such that $(W \times U) \cap \mathrm{Gr}\,(F^*) = \emptyset$. Let n be such that $\Phi(x_0) \subset U_n(x_0) \subset U$. Put $W' := W_n(x_0) \cap W$. This set is non-empty (it contains x_0) and, moreover, since the couple $(W_n(x_0), V_n(x_0))$ is admissible then the couple $(W', V_n(x_0))$ is admissible too. This means that $F(W') \cap V_n \neq \emptyset$ and, therefore, $(W' \times V_n) \cap \mathrm{Gr}\,(F^*) \neq \emptyset$. On the other hand, $W' \times V_n \subset W \times U$. This is a contradiction.

Put now $G(x) := \Phi(x) \cap F^*(x)$, $x \in X_1$. The mapping G, which acts beteen X_1 and T_s, is obviously a non-empty valued selection of F^* on X_1. As an usco mapping Φ has a closed graph. Since the mapping G is an intersection of two mappings with closed graphs, it has a closed graph too. On the othe hand, as we already mentoned, every non-empty valued mapping with closed graph which is contained in an usco mapping is usco itself. Hence, G is usco.

Finally, if the strategy s is such that every s-play has a one-point target set, then obviously the mapping Φ is single-valued and continuous, and therefore, the same is true for the mapping G. Moreover, if F is non-empty valued and F and s are strictly coordinated then for any $x \in X_1$ the set $\Phi(x)$ (which is a singleton) belongs also to $\bar{F}(x)$. Indeed, suppose $\Phi(x_0) \notin \bar{F}(x_0)$ for some $x_0 \in X_1$. Then for some open set V of Y we have $\Phi(x_0) \subset V$ and $\bar{F}(x_0) \cap V = \emptyset$. Let n_0 be so that $V_{n_0}(x_0) \subset V$. But the couple $(W_{n_0}(x_0), V_{n_0}(x_0))$ is strictly admissible, which is a contradiction with $\bar{F}(x_0) \cap V = \emptyset$. Therefore, the mapping $G(x) = \Phi(x) \in \bar{F}(x)$, $x \in X_1$ is a single-valued and conyinuous selection of $\bar{F}$ on X_1. The proof of Theorem 4.3 is completed. ■

Remark 4.7 The special case when $F = f^{-1}$ for some continuous and single-valued mapping $f\colon Y \to X$ deserves more attention. It was essentially considered by E. Michael in [M5] (see Theorem 5.9 below). It can be shown in this case, that, in addition to the conclusion of the above theorem, the set $C\colon = G(X_1)$ is a G_δ-subset of Y. Indeed, with the notations in the proof above, let $C_n\colon = \cup\{f^{-1}(W) \cap V\colon (U, V, W) \in \Gamma_n$ for some uniquely determined $U, V\}$. The continuity of f implies that every C_n is open in Y. Since for every $x \in X_1$ we have $F(\cap_{n=0}^{\infty} U_n(x)) = f^{-}(\cap_{n=0}^{\infty} U_n(x)) = \cap_{n=0}^{\infty} f^{-1}(U_n(x))$ it is easily verified that $C = \cap_{n=0}^{\infty} C_n$.

Further, Theorem 4.5 can be proved using exactly the same scheme (starting with a lemma similar to Lemma 4.6) replacing the sets $F^{-1}V$, $V \subset Y$, by the sets $F^{\#}V$. Indeed, given $F : X \to Y$, call a couple (W, V) of open sets W of X and V of Y *strongly admissible* if $\mathrm{Cl}\,(F(W)) \subset V$. The next lemma is proved exactly in the same way as Lemma 4.6 (using also the regularity of Y).

Lemma 4.8 *In the assumptions of Theorem 4.5, let $(U_1, V_1, \ldots U_n, V_n)$ be a partial s-play and W_n be a non-empty open subset of X such that (W_n, V_n) is a strongly admissible pair. Suppose that $\Gamma(W_n) = \{(W, U, V)\}$ is a maximal system of triples such that:*

(i) U is a non-empty open subset of V_n;

(ii) $V = s(U_1, V_1, \ldots, U_n, V_n, U)$;

(iii) (W, V) is a strongly admissible pair, so that $W \subset W_n$;

(iv) $\gamma(W_n) := \{W : (W, U, V) \in \Gamma(W_n)\}$ is a pairwise disjoint system of open subsets of W_n.

Then the set $H(W_n) := \cup\{W : W \in \gamma(W_n)\}$ is dense in W_n.

Again the lemma is true for $n = 0$ if we put $W_0 = X$, $U_0 = V_0 = Y$ because the couple (X, Y) is strongly admissible.

Proof of Theorem 4.5. The proof of this theorem follows the same pattern as of Theorem 4.3 using Lemma 4.8. As above we obtain a dense G_δ-subset X_1 of X so that each $x \in X_1$ uniquely determines a sequence of triples $(W_n(x), U_n(x), V_n(x))_{n=0}^{\infty}$ such that $x_0 \in \cap_{n=0}^{\infty} W_n(x)$, $(W_n(x), V_n(x))$ is a strongly admissible pair for every n and the sequence $\{U_n(x), V_n(x)\}_{n=1}^{\infty}$ is an s-play.

Again as above the mapping $\Phi(x) := \cap_{n=1}^{\infty} U_n(x) = \cap_{n=1}^{\infty} V_n(x)$, $x \in X_1$ is un usco mapping from X_1 to T_s. Similarly (using also that $\mathrm{Dom}\,(F)$ is dense in X) one sees that for every $x \in X_1$ is true $\Phi(x) \cap F^*(x) \neq \emptyset$. What is true in addition here is that $F^*(x) \subset \Phi(x)$ for any $x \in X_1$. Indeed, for every such x, by definition $F^*(x) \subset \mathrm{Cl}\,(F(W_n(x))) \subset V_n(x)$ for every n whence the desired inclusion. Since F^* is with a closed graph, the latter entails that the restriction $F^*|X_1 : X_1 \to T_s$ is usco.

Now, let the strategy s be so that for each s play the target set is a singleton. By the above, this means that we have $F^*(x) = \Phi(x)$ for every $x \in X_1$ and the latter set is a singleton. We prove that F^* is not only single-valued but also upper-semicontinuous at any $x \in X_1$ as a mapping from X into T_s. To this end let $F^*(x) = \Phi(x) \subset V$ for some open V and $x \in X_1$. There is some n so that $V_n(x) \subset V$. Let $x' \in W_n(x)$. Then (since $(W_n(x), V_n(x))$ is strongly admissible), $F^*(x') \subset \mathrm{Cl}\,(F(W_n(x))) \subset V_n(x) \subset V$.

Finally, if the mapping F is non-empty valued one sees as in the proof of the previous theorem that $\bar{F}(x) \cap \Phi(x) \neq \emptyset$ for every $x \in X_1$. Now, using the same argument as above $\bar{F}$ coincide with Φ at the points of X_1 (whence is single-valued at these points) and is upper semicontinous at any $x \in X_1$ as a mapping from X to T_s. The proof of the theorem is completed. ■

Remark 4.9 We would like to mention a variant of the above theorem related to another modification of the notion of a strategy for the player α in the game $BM(X)$. Namely, let the strategy s for the player α in the space Y be so that for each s-play $p = \{U_i, V_i\}_{i=1}^{\infty}$ the target space $T(p)$ is either empty or if it is non-empty then the partial play is complete. Then, if we assume the hypothesis of Theorem 4.5 with this new type of strategy (and even without assuming the denseness of $\mathrm{Dom}\,(F)$), its conclusion is changed to: for every $x \in X_1$ the mapping F^* (resp. $\bar{F}$) is either empty, or F^* (resp. $\bar{F}$) is usco (or upper semicontinuous and single-valued) at x. We will comme across with strategies like this in the applications. Moreover, a closed look at the proof shows that the latter result is true even without supposing the openess of the sets U_n, V_n (i.e. we may allow a very general game in which the players choose just non-empty subsets and to keep only the requirement for the strategy s to have either empty target set or a one point set for which the corresponidng play is complete).

5 Sufficient conditions for the existence of residually defined selections of set-valued mappings

In this section we will give (rather general) conditions for the assumptions of Theorems 4.3, 4.5 to be fulfilled. Before that we need some more notions.

We start with a further weakening of the notion of lower semicontinuity. Let F be a set-valued map between the topological spaces X and Y. The mapping F is said to be *lower demicontinuous* at some $x_0 \in X$ if for every open V of Y with $F(x_0) \cap V \neq \emptyset$, there is some open set U of X with $x_0 \in \mathrm{Cl}\,(U)$ and the set $F^{-1}(V)$ is dense in U. F is lower demicontinuous in X if it is lower demicontinuous at any $x \in X$. It is straitforward to verify that F is lower demicontinuous in X if for every open V in Y the set $\mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V)$ is dense in $\mathrm{Cl}\,F^{-1}(V)$. Obviously, every lsc in X mapping $F : X \to Y$ is lower demicontinuous in X. Observe that, by Proposition 3.1 every minimal closed graph and upper quasi continuous at any point of its domain mapping $F : X \to Y$ is lower demicontinuous in X.

Call a set-valued mapping $F : X \to Y$ *demi-open* (as in [HeS] for single-valued maps) if for every open set U in X the set $\operatorname{Int}\operatorname{Cl} F(U)$ is dense in $\operatorname{Cl} F(U)$. The following is elementary:

Proposition 5.1 *The mapping $F : X \to Y$ is lower demicontunous in X if, and only if, the set-valued map $F^{-1} : Y \to X$ is demi-open.*

Here is a situation when we have the assumptions of Theorem 4.3 fulfilled.

Proposition 5.2 *Let $F : X \to Y$ be a lower demicontinuous and demi-open mapping and s be a strategy for the player α in the game $BM(Y)$. Then F and s are coordinated. If, in addition, F is non-empty valued and F is lower quasicontinuous, then F and s are strictly coordinated*

Proof: Suppose that F is lower demicontinuous and demi-open and s is a strategy for α in the game $BM(Y)$. Let $U_1 \supset V_1 \supset \cdots U_n \supset V_n$ be a partial s-play, i.e. $V_i = s(U_1, V_1, \ldots, U_i)$ for every $i = 1, 2, \ldots n$. We have to prove that the set $\cup\{\operatorname{Int}\operatorname{Cl} F^{-1}(V) : V = s(U_1, V_1, \ldots, U_n, V_n, U)$ for some non-empty open $U \subset V_n\}$ is dense in $\operatorname{Int}\operatorname{Cl} F^{-1}(V_n)$.

Let W be a non-empty open subset of X such that $W \cap \operatorname{Int}\operatorname{Cl} F^{-1}(V_n) \neq \emptyset$. Since F is demi-open and $F(W) \cap V_n \neq \emptyset$ there exists some non-empty open set $U \subset V_n \cap \operatorname{Int}\operatorname{Cl} F(W)$. Let $V = s(U_1, V_1, \ldots, U_n, V_n, U) \subset U$. Since $V \cap F(W) \neq \emptyset$ and F is lower demicontinuous in X we have that $W \cap \operatorname{Int}\operatorname{Cl} F^{-1}(V) \neq \emptyset$. Thus F and s are coordinated.

Assuming in addition that F is non-empty valued and lower quasicontinuous in X, the proof that F and s are strictly coordinated (i.e. that the set $\cup\{\operatorname{Int} F^{-1}(V) : V = s(U_1, V_1, \ldots, U_n, V_n, U)$ for some non-empty open $U \subset V_n\}$ is dense in the set $F^{-1}(V_n)$) follows exactly the same pattern. This completes the proof. ■

Now we look for conditions assuring the existence of complete winning strategies for the player α in the Banach-Mazur game. First, let us remind that the completely regular topological space Y is Čech complete if it lies as a G_δ-subset in its Stone-Čech compactification βX (or in any other compactification of Y). Recall also that a *cover* δ for a topological space Y is a family of subsets of Y whose union is Y. The cover is *open* if it consists of open sets. Given two covers δ and γ of X we write $\delta \subset \gamma$ when every element of δ is contained in some element of γ.

The next proposition gives sufficient conditions for a space Y to admit a complete strategy as defined above. In fact, as it will be seen by Theorems 11.1, 11.2 in the next chapter, these conditions are also necessary.

Proposition 5.3 *Let Y be a completely regular topological space which contains a dense subset Y_1 which is Čech complete in the induced topology. Then Y possesses* *a complete α-winning stationary strategy s in the game $BM(Y)$ so that $T_s \subset Y_1$. If, in addition the space Y_1 is completely metrizable, then the strategy s can be taken in such a way that for every s-play $p = \{U_i, V_i\}_{i=1}^{\infty}$ the target set $T(p)$ is a singleton.*

Proof: By a characterization of the Čech completeness [Fro1, Fro2], the space Y_1 admits a countable family $\{\delta_n\}_{n=1}^{\infty}$ of open covers of Y_1 so that if $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of integers and $\{U_{n_k}\}_{k=1}^{\infty}$ is a nested family so that $U_{n_k} \in \delta_{n_k}$ for every k, then the sequence $\{U_{n_k}\}_{k=1}^{\infty}$ is complete in Y_1. In fact, since Y_1 is dense in Y and Y is completely regular, we may think that each δ_n consists of open sets of Y and if $\{U_{n_k}\}_{k=1}^{\infty}$ is a nested family as above, then the sequence $\{U_{n_k}\}_{k=1}^{\infty}$ is complete in Y as well. Moreover, for every such sequence $\cap_{k=1}^{\infty} U_{n_k} \subset Y_1$.

Suppose (without loss of generality) that $\delta_1 := \{Y\}$ and for any $U \subset Y$ define $d(U) := \max\{n \geq 1 : \text{there is some } W \in \delta_n \text{ with } U \subset W\}$ (it is possible that for some U, $d(U) = \infty$). Now, let τ be the topology in Y and set $\tau_0 := \tau \setminus \{\emptyset\}$. Given a non-empty open set U of Y let us consider the family

$$\gamma_U := \begin{cases} \{W \in \tau_0 : \mathrm{Cl}\,(W) \subset U\} & \text{if } d(U) = \infty, \text{ and} \\ \{W \in \delta_{2d(U)} : W \subset U\} & \text{otherwise.} \end{cases}$$

Since Y is completely regular and the union of the sets from each δ_n is dense in Y then γ_U is a non-empty family for every $U \in \tau_0$. Fix some mapping $t : \tau_0 \to \cup\{\gamma_U : U \in \tau_0\}$ with the property $t(U) \in \gamma_U$ for each $U \in \tau_0$.

The so defined mapping t is a complete α-winning (stationary) strategy in the game $BM(Y)$. Indeed, let $\{U_i\}_{i=1}^{\infty}$ be a family of non-empty open sets of Y so that $U_{i+1} \subset t(U_i)$ for every $i = 1, 2 \ldots$. We have two possibilities: either for some i $d(U_i) = \infty$ or the sequence $\{d(U_i)\}$ is a strictly increasing sequence of integers. In the first case we may suppose $i = 1$ and we see that the sequence $\{U_i\}$ lies in some compact subspace of Y of the type $\cap_{n=1}^{\infty} W_n$ with $W_n \in \delta_n$ for each n. Because $\mathrm{Cl}\,(U_{i+1}) \subset U_i$ for every i we easily see that the sequence $\{U_i\}$ is complete. In the second case $t(U_i) \in \delta_{2d(U_i)}$ for every i and hence the sequence $\{t(U_i)\}$ (as well as $\{U_i\}$) is complete. Observe that in both cases $T_s \subset Y_1$.

If in addition Y_1 is metrizable by some complete metric ρ we may think that the subsets from δ_n have diameters (with respect to ρ) less than $1/n$ for every $n = 1, 2 \ldots$. In this case the strategy t will have the required additional property. The proof of the proposition is completed. ■

The following theorem is an immediate corollary from Theorem 4.3 and the above proposition.

Theorem 5.4 *Suppose X is a Baire space and Y is a topological space which contain a dense Čech complete subspace Y_1. Let $F : X \to Y$ be a demi-open set-valued mapping with dense domain. Suppose that either F is lower demicontinuous mapping with closed graph, or F is lower quasicontinuous and has non-empty closed images.*

Then there exists a dense G_δ-subset X_1 of X and a non-empty-valued usco mapping $G : X_1 \to Y_1$ such that:

a) $X_1 \subset \mathrm{Dom}\,(F)$;

b) $G(x) \subset F(x)$ for every $x \in X_1$, i.e. G is a selection of F on X_1.

If, in addition, Y_1 is completely metrizable, then the selection G is single-valued.

Further, we see that if the very space Y is Čech complete (or completely metrizable), then the conclusion of the above theorem can be obtained without assuming demi-openess of the mapping.

Proposition 5.5 *Let Y be Čech complete. Then there exists an α-winning strategy s in Y which is coordinated with any lower demicontinuous mapping $F : X \to Y$ and strictly coordinated with any non-empty valued lower quasicontinuous mapping $F : X \to Y$. If Y is a complete metric space, then the strategy s has the additional property that for any s-play $p = \{U_i, V_i\}$ the target set $T(p)$ is a singleton.*

Proof: Let $\{\delta_k\}_{k=1}^{\infty}$ be a countable family of open covers of Y as in the proof of Proposition 5.3 which exists according to the Frolík characterization of the Čech completyeness [Fro1, Fro2]. Again as in the same proposition we may think that if Y is a complete metric space them the covers are such that for every k each element of δ_k has diameter less than $1/k$. Suppose the strategy s is constructed up to the k-th step, $k \geq 0$ (if $k = 0$ we put $U_0 = V_0 = Y$). Let $U_1 \supset V_1 \supset \cdots \supset U_k \supset V_k$ be a partial play with length k obtained by the constructed strategy s and let $U \subset V_k$. If U is such that $U \subset \delta_{k+1}$ then we define s on the $k+1$ step by putting $s(U_1, V_1, \ldots, U_k, V_k, U) := U$. For all other open sets U contained in V_k we define the set $s(U_1, V_1, \ldots, U_k, V_k, U)$ to be any non-empty open subset V of U such that $V \in \delta_{k+1}$ and $\mathrm{Cl}\,(V) \subset U$.

It is clear that the so constructed strategy is a complete α-winning one. We show that it is coordinated with any lower demicontinous mapping $F : X \to Y$ (resp. strictly coordinated with F if the latter is non-empty valued and lower quasicontinuous). Indeed, take some s-partial play $U_1 \supset V_1 \supset \cdots \supset U_k \supset V_k$ and some non-void open $H \subset \mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_k)$. Take some $x_0 \in H \cap F^{-1}(V_k)$ and $y_0 \in F(x_0) \cap V_k$. Since δ_{k+1} is a cover of Y there is some $U \in \delta_{k+1}$ so that $y_0 \in U$. Since F is lower demicontinuous and $H \cap F^{-1}(U) \neq \emptyset$ we have that $H \cap \mathrm{Int}\,\mathrm{Cl}\,F^{-1}(U) \neq \emptyset$. On the other hand, $U = s(U_1, V_1, \ldots, U_k, V_k, U)$ and this shows that F and s are coordinated. The verification that F and s are strictly coordinated provided F is non-empty valued an lower quasicontinuos is completely similar. This completes the proof. ■

Having this proposition, the following theorem is again an immediate corollary from Theorem 4.3.

Theorem 5.6 *Let $F : X \to Y$ be a set-valued mapping between the Baire space X and the Čech complete space Y. Suppose that either F is lower demicontinuous with dense domain and closed graph, or F is lower demicontinuous and has non-empty closed images. Then there exists a dense and G_δ-subset X_1 of X and a non-empty-valued usco mapping $G : X_1 \to Y$ such that:*

a) $X_1 \subset \mathrm{Dom}\,(F)$;

b) G is a selection of F on X_1.

If Y is a complete metric space then the selection G is single-valued.

We give now some examples showing that the assumptions on F in the above theorems are essential.

Example 5.7 *Let the segment $X := [0,1]$ be endowed with the usual topology on the real line. Let further $B \subset [0,1]$ be such that both B and $X\backslash B$ are dense Baire subspaces of X (e.g. let B be a Bernstein subset of $[0,1]$, see [En], 4.5.5 (b) and 5.5.4). Put $Y\colon = (B\times\{0\})\cup((X\backslash B)\times\{1\})\cup(X\times(0,1))$, where $(0,1)$ is the open interval, and consider on Y the topology inherited by the product topology of $X\times X$. Let $F\colon X\to Y$ be defined by $F(x) = (x,0)$ if $x\in B$ and $F(x) = (x,1)$ if $x \in X\backslash B$. F has a closed graph because $B\times\{0\}$ and $(X\backslash B)\times\{1\}$ are closed subsets of Y. F is obviously lower demicontinuous. But there is no dense G_δ-subset of X on which F possesses a continuous selection. In this example F is not coordinated with any complete α-winning strategy in Y. The reason is that F is not demi-open.*

Example 5.8 *Let X and B be as in the previous example. Define $F\colon X\to\{0,1\}$ by $F(x) = 1$ if $x\in B$ and $F(x) = 0$ otherwise. F is lower demicontinuous (but not lower quasicontinuous) and F does not have a continuous selection on a dense G_δ-subset of X. Here $\mathrm{Gr}\,(F)$ is not closed.*

We give some further consequences of Theorem 5.4. Let us start rt with a result of E. Michael [M5].

Theorem 5.9 ([M5], Theorem 7.2) *Let $f\colon Y\to X$ be a continuous and demi-open single-valued mapping acting from the regular space Y which contain a dense Čech complete subspace into the space X such that $f(Y)$ is dense in X. Then there exist a G_δ-subset C of Y and a dense G_δ-subset D of X such that $f|C\colon C\to D$ is perfect and onto.*

Proof: Let $F\colon = f^{-1}\colon X\to Y$. By assumptions $\mathrm{Dom}\,(F) = f(Y)$ is dense in X. It can be seen that (e.g. [M5], Proposition 6.6) X is a Baire space. Since f is demi-open then by Proposition 5.1 F is lower demicontunous. On the other hand, the continuity of f implies that F is open and has closed graph. Therefore, by Theorem 5.4 there are a dense G_δ-subset D of X and an usco $G\colon D\to Y$ such that

$D \subset \mathrm{Dom}\,(F) = f(Y)$ and G is a selection of F. Let $C := G(D)$. Since $F = f^{-1}$ and f is single-valued then $F(x_1) \cap F(x_2) = \emptyset$ for $x_1 \neq x_2$. Hence $G(x) = F(x) \cap C$ for every $x \in D$. This implies that $G = (f|C)^{-1}$ showing that $f|C \colon C \to D$ is perfect and onto. The rest follows by Remark 4.7.

Theorem 5.10 ([M5], Theorem 7.3) *If the space Y in Theorem 5.9 contains a dense and completely metrizable subspace then $f|C$ can be considered to be a homeomorphism.*

Proof: As above using again Theorem 5.4.

Remark 5.11 Let us note that Theorems 7.2 and 7.3 from the paper of E. Michael [M5] contain additional information about the sets C and D (special completeness properties). This properties do not follow directly from our Theorems 5.4.

In connection with the above theorem E. Michael raised the following questions in [M5], Question 7.4: Let f be an open and continuous single-valued mapping from a Čech complete space Y onto a regular (or even metrizable) space X. Must f map some non-empty subset C of Y homeomorphically onto a dense G_δ-subset of X. The answer when X Čech complete but not metrizable is negative.

Example 5.12 *Let τ be a cardinal which is greater or equal to the first uncountable cardinal. By a result of Pasinkov [Pas], Theorem 2, there are a compact space Y_τ of weight τ with $\dim Y_\tau = 1$, and a continuous and open mapping f which maps Y_τ onto $[0,1]^\tau$ and such that $\dim(f^{-1}(x)) = 0$ for every $x \in [0,1]^\tau$. Let us observe that if H is a non-empty G_δ-subset of $[0,1]^\tau$ then H contains a homeomorphic image of $[0,1]^\tau$. Suppose now that there is $C \subset Y_\tau$, $C \neq \emptyset$, such that $f(C) = H$, H is G_δ in $[0,1]^\tau$ and $f|C$ is a homeomorphism. Then C contains a homeomorphic image of $[0,1]^\tau$, so the same does Y_τ. But $\dim[0,1]^\tau = \infty$ while $\dim Y_\tau = 1$. This is a contradiction.*

Now, we show that when the range space Y is second countable the conclusion of Theorem 5.4 can be obtained under weakened assumptions on the mapping F.

Theorem 5.13 *Let F be a mapping with closed graph acting from a Baire space X into the completely metrizable separable space Y. Let $\mathrm{Dom}\,(F) = X$. Then there exist a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f \colon X_1 \to Y$ which is a selection of F on X_1.*

Proof: Let $\{V_n\}$, $n \geq 1$, be a countable base for the topology in Y. Consider the sets $H_n = \mathrm{Cl}\, F^{-1}(V_n) \backslash \mathrm{Int}\,\mathrm{Cl}\, F^{-1}(V_n)$. The sets H_n are closed and nowhere dense in X and consequently the set $X' \colon = \cap_{n=1}^{\infty} X \backslash H_n$ is dense G_δ in X. Further, since

$\{V_n\}$, $n \geq 1$ is a base in Y, it is a routine matter to check that the restriction $F|X'$ of the mapping F on X' is lower demicontinous. To complete the proof, apply Theorem 5.4 for the mapping $F|X' \colon X' \to Y$.

Finally, we will give several applications of our second main selection theorem and will prove results in which a mapping $F \colon X \to Y$ is usco (or single-valued and usc) itself at the points of a residual subset of its domain. We start with a general result when the range space is metric.

A mapping $F \colon X \to Y$, where Y is a metric space with a metric ρ, is said to be *fragmented* by ρ ([HaJT], p.217) if for every $\varepsilon > 0$ and every non-empty open set U in X there exists a non-empty open $U' \subset U$ such that $\rho - \operatorname{diam}(F(U')) < \varepsilon$.

Theorem 5.14 *Let $F \colon X \to Y$, where X is a Baire space and (Y, ρ) is a metric space, be fragmented by the metric ρ. Then there exists a dense G_δ-subset X_1 of X such that:*

a) for every $x \in X_1$ either $F(x) = \emptyset$ or F is single-valued and usc at x.

b) if ρ is complete, $\operatorname{Dom}(F)$ *is dense in X and F has a closed graph then $X_1 \subset$* $\operatorname{Dom}(F)$ *and at the points of X_1 F is single-valued and usco.*

Proof: Put $U_0 = V_0 = Y$ and define the stratgey s as above: given a partial play $\{U_i, V_i\}_{i=0}^{k}$, $k \geq 0$, and a non-empty open $U \subset V_k$, let $V := s(U_0, V_0, \ldots, U_k, V_k, U)$ be a non-empty open subset of Y so that $\operatorname{Cl}(V) \subset U$ and $\operatorname{diam}(V) < 1/(k+1)$. It is easily verified that, if X is a complete metric space then s is complete and in the general case, s satisfies the weaker condition from Remark 4.9. The so constructed strategy s and the mapping F are strongly coordinated because of the fragmentability of F. Therefore, the conclusion of the theorem follows by Theorem 4.5 and Remark 4.9. ■

The next applications concerns minimal mappings. First, we show the fundamental fact that for such mappings the existence of a densely defined selection in fact implies the coincidence between the mapping and its selection and even more.

Theorem 5.15 *Let $F : X \to Y$ satisfy condition c) from Proposition 3.1 (i.e.* $\operatorname{Int} F^{\#}(V)$ *is dense in $F^{-1}(V)$ for every open V in Y). Let F possess an usco selection $G : X_1 \to Y$ on some dense subset X_1 of X. Then F coincides with G at the points of X_1 and, moreover, F is usc at any $x \in X_1$.*

Proof: Suppose that $F(x_0) \setminus G(x_0)$ for some $x_0 \in X_1$. Then (because $G(x_0)$ is compact) there are some point y_0 from this set difference and non-empty open sets V_0 and V_1 of Y with $y_0 \in V_0$, $G(x_0) \subset V_1$ and $V_0 \cap V_1 = \emptyset$. By the fact that G is ucs there is some open set U of X containing x_0 with the property $G(X_1 \cap U) \subset V_1$. On the other hand, by the property of the mapping F there is some non-empty $U_1 \subset U$ with $F(U) \subset V_0$. This is a contradiction since $U' \cap X_1 \neq \emptyset$ and G is a selection of

F in X_1. Therefore, $F(x) = G(x)$ for any $x \in X_1$. To finish the proof, we show that at the points of X_1, F is usc as a mapping from X into Y.

Indeed, take $x_0 \in X_1$ and let V be an open subset of Y such that $F(x_0) \subset V$. Since $F(x_0)$ is compact and Y is regular there is an open subset W of Y with $F(x_0) \subset W$ and $\mathrm{Cl}\,(W) \subset V$. Having in mind that F is usc in X_1 (since $F = G$ on this set) there is an open subset U of X such that $F(x) \subset W$ for every $x \in U \cap X_1$. Suppose there is a point $x_1 \in U \cap \mathrm{Dom}\,(F)$ such that $F(x_1) \setminus \mathrm{Cl}\,(W) \neq \emptyset$. Take y_0 from this latter set. Then there is an open in Y set V_0 such that $y_0 \in V_0$ and $V_0 \cap \mathrm{Cl}\,(W) = \emptyset$. Since F satisfies Proposition 3.1 c) we have that $\mathrm{Int}\, F^{\#}(V_0)$ is dense in $F^{-1}(V_0)$. Therefore, there exists a non-empty open $U' \subset U$ such that $F(U') \subset V_0$. This is a contradiction since $U' \cap X_1$ is a non-empty subset of $U \cap X_1$. The proof is completed. ∎

Having the above fact, the following theorem is an immediate consequence from Theorem 4.3.

Theorem 5.16 *Let F be a minimal upper quasicontinuous and closed graph mapping acting from a Baire space X into the regular space Y such that* $\mathrm{Dom}\,(F)$ *is dense in X. Suppose that F is coordinated with some complete α-winning strategy s in Y. Then there exists a dense G_δ-subset X_1 of X such that $X_1 \subset \mathrm{Dom}\,(F)$, F maps X_1 into T_s and at the points of X_1 F is usco. If the strategy s is such that each s play has a one-point target set, then F is also single-valued at the points of X_1.*

We already mentioned that sometimes set-valued mappings that satisfy condition c) from Proposition 3.1 are called simply minimal. For such mappings, the conclusion of the above theorem can be obtainde without assuming the closedness of the graph, provided the mapping is non-empty valued. Namely, we have the following theorem

Theorem 5.17 *Let F be a non-empty valued set-valued mapping acting from a Baire space X into the regular space Y which satisfies condition c) from Proposition 3.1. Suppose that F is coordinated with some complete α-winning strategy s in Y. Then there exists a dense G_δ-subset X_1 of X such that $X_1 \subset \mathrm{Dom}\,(F)$, F maps X_1 into T_s and at the points of X_1 F is usco. If the strategy s is such that each s play has a one-point target set, then F is also single-valued at the points of X_1.*

Proof: Observe that if F is non-empty valued and satsifies condition c) from Proposition 3.1 then F is lower quasicontinuous. The conclusion then follows by Theorem 4.5, Proposition 5.2 and the above argument. ∎

6 Some applications to the geometry of Banach spaces

In this and the next section we see how the results about residually defined selections of set-valued mappings can be applied to get various results in differet fileds. Another applications of the selection theorems to optimization will be given in Sections 10–11.

Let $(E, \|\cdot\|)$ designate a real Banach space with norm $\|\cdot\|$. By E^*, as usual, we will denote the dual of E endowed with the standard dual norm $\|x^*\| = \sup\{|\langle x, x^*\rangle| \colon x \in B\}$, $x^* \in E^*$, where $B \colon = \{x \in E \colon \|x\| \leq 1\}$ is the closed unit ball in E and $\langle \cdot, \cdot \rangle$ designates the usual duality between E and E^*. The weak topology in E and weak star topology in E^* will be denoted by w and w^* respectively.

Let $f : E \to \mathbb{R}$ be a continuous convex function. The *subdifferential* ∂_f of f, is a multivalued mapping acting from E into E^*, defined by the formula:

$$\partial_f(x) \colon = \{x^* \in E^* \colon \langle y - x, x^* \rangle \leq f(y) - f(x) \text{ for every } y \in E\}, \quad x \in E.$$

It easily follows by this definition that ∂_f is a *monotone mapping*, i.e. for every $x_1, x_2 \in E$ and for every $x_1^* \in \partial_f(x_1)$ and $x_2^* \in \partial_f(x_2)$ one has $\langle x_1 - x_2, x_1^* - x_2^* >\geq 0$.

The following facts are well-known and may be found in [Ph].

i) $\partial_f \colon E \to E^*$ is a norm-to-w^* usco mapping with $\mathrm{Dom}\,(\partial_f) = E$;

ii) f is Gâteaux differentiable at $x_0 \in E$ iff $\partial_f(x_0)$ is a singleton;

iii) f is Fréchet differentiable at $x_0 \in E$ iff ∂_f is single-valued and norm-to-norm usc at x_0.

In 1933 Mazur [Ma] proved that if E is a separable Banach space, then f is Gâteaux differentiable at the points of some residual subset of E (in such a case we say that f is generically Gâteaux differentiable). We will see how the theorems from the previous paragraph can be used to show generic differentiability of convex functions in some other cases.

Let $d(\cdot, \cdot)$ be a metric in E^*. It is said that *d fragments E^** ([JaRo]) if for every bounded subset A of E^* and every $\varepsilon > 0$ there exists some w^*-open subset V of E^* such that $V \cap A \neq \emptyset$ and $d - \mathrm{diam}\,(V \cap A) < \varepsilon$. It is well-known that E^* has Radon-Nikodym Property (RNP) iff E^* is fragmented by the metric generated by the dual norm in E^*. Another important example is provided in the paper of Ribarska [Ri]: if the norm in E is Gâteaux differentiable at each $x \in E$, $x \neq 0$, then E is fragmented by some metric d.

Theorem 6.1 *Let E be a Banach space with dual E^* fragmented by some metric d and $f \colon E \to \mathbf{R}$ be a continuous convex function. Then there exists a dense G_δ-subset D of E at the points of which f is Gâteaux differentiable. If the metric d is generated by the dual norm in E^* then at the points of D f is Fréchet differentiable.*

Proof: Since $\partial_f\colon (E, \|\cdot\|) \to (E^*, w^*)$ is usco there exists some minimal usco $F\colon (E, \|\cdot\|) \to (E^*, w^*)$ which is a selection of ∂_f. The following lemma gives us the possibility to apply Theorem 5.14.

Lemma 6.2 *The mapping $F\colon (E, \|\cdot\|) \to (E^*, d)$ is fragmented by the metric d.*

Proof of the lemma: Consider the closed unit ball B^* in E^*. Since F is norm-to-w^* usco the sets $F^{-1}(nB^*)$ are closed in X for every $n \geq 1$. Take an arbitrary non-empty open set U of E and positive ε. Observe that $U \subset \cup_{n=1}^{\infty} F^{-1}(nB^*)$. By the Baire theorem $U_1 := U \cap \operatorname{Int} F^{-1}(nB^*) \neq \emptyset$ for some $n \geq 1$. Since F is minimal we get $F(U_1) \subset nB^*$ (Proposition 3.1). On the other hand, d fragments E^*, hence there exists a w^*-open set V of E^* such that $V \cap F(U_1) \neq \emptyset$ and $d-\operatorname{diam}(V \cap F(U_1)) < \varepsilon$. Again by the minimality of F (Proposition 3.1 c)) one gets a non-empty open $U' \subset U_1$ such that $F(U') \subset V$. Therefore $d - \operatorname{diam}(F(U')) < d - \operatorname{diam}(V \cap F(U_1)) < \varepsilon$. The proof of the lemma is completed. ■

Now, let us go back to the proof of Theorem 6.1. By Theorem 5.14 there exists a dense G_δ-subset D of E at the points of which F is single-valued and norm-to-d usc. We will show also that

i) $\partial_f(x) = F(x)$ for every $x \in D$ (i.e. F is Gâteaux differentiable at the points of D);

ii) if the metric d is generated by the dual norm in E^* then ∂_f is single-valued and norm-to-norm u.s.c. at any $x \in D$ (this would imply that f is Fréchet differentiable at the points of D).

We prove ii). Let $x_0 \in D$ and $\varepsilon > 0$. Then $F(x_0) = \{x_0^*\}$ for some $x_0^* \in E^*$ and $x_0^* \in \partial_f(x_0)$. Since F is norm-to-norm usc at x_0 there exists some open U of X such that $x_0 \in U$ and $F(U) \subset x_0^* + \varepsilon B^*$. It suffices to show that $\partial_f(U) \subset x_0^* + \varepsilon B^*$. Suppose this is not the case and take some $x_1^* \in \partial_f(x_1) \backslash \{x_0^* + \varepsilon B^*\}$ where $x_1 \in U$. Then there exists $h \in E, \| h \| = 1$, which strongly separates x_1^* from the close convex ball $x_0^* + \varepsilon B^*$, i.e. for some $\delta > 0$ the w^*-open set $H_\delta \colon = \{x^* \in E^* : \langle h, x^* \rangle > \langle h, x_1^* \rangle - \delta\}$ does not intersect $x_0^* + \varepsilon B^*$.

Consider, for $t > 0$, the point $x(t) \colon = x_1 + th$. By the monotonicity of ∂_f we have

$$0 \leq \langle x(t) - x_1, x^* - x_1^* \rangle = t\langle h, x^* - x_1^* \rangle \text{ for every } x^* \in \partial_f(x(t)).$$

This means that $\partial_f(x(t)) \subset H_\delta$ for every $t > 0$. However, when t is small enough $x(t) \in U$ and hence $F(x(t)) \subset x_0^* + \epsilon B^*$. This is a contradiction since F is a selection of ∂_f. The proof of ii) is completed. The proof of i) is simpler. ■

In the next theorem the Banach space E is identified with its natural embedding in its second dual E^{**}.

Theorem 6.3 *Let E be a separable Banach space and $f\colon E^* \to \mathbb{R}$ be a continuous convex function such that the set $A\colon = \{x^* \in E^*\colon \partial_f(x^*) \cap E \neq \emptyset\}$ is residual in the norm topology in E^*. Then f is Fréchet differentiable on a dense G_δ-subset A' of E^* and $\partial_f(x^*) \in E$ for every $x^* \in A'$.*

Proof: Consider the mapping $F\colon A \to E$ defined by $F(x^*)\colon = \partial_f(x^*) \cap E$, $x^* \in A$. Evidently $\mathrm{Dom}\,(F) = A$. On the other hand, since ∂_f has a closed graph, the mapping F has a closed graph too. Hence, by Theorem 5.13 there exist a residual subset A' of A and a continuous single-valued mapping $h : A' \to E$ which is a selection of F. Hence A' is residual in E^*. Arguments similar to the used in the proof of ii) from Theorem 6.1 show that for every $x^* \in A'$ $\partial_f(x^*) = h(x^*)$ and ∂_f is norm-to-norm usc at x^*. This means that f is Fréchet differentiable at the points of A'.

Put, further, $BP\colon = \{x^* \in E^* : x^*$ attains its maximum on the unit ball $B\}$. According to the famous Bishop-Phelps theorem [BiPh], the set BP is dense in the norm topology in E^*.

Corollary 6.4 *Let E be a separable Banach space such that the Bishop-Phelps set BP is residual in E^*. Then the norm in E^* is Fréchet differentiable at the points of some residual subset A' of E^* and for every $x^* \in A'$ $\partial_{\|\cdot\|}(x^*) \in B$. In particular, for every $x^* \in A'$ the maximization problem (B, x^*) is Tikhonov well-posed.*

Proof: Let B^{**} be the closed unit ball in E^{**}. It is well known that $\partial_{\|\cdot\|}(x^*) = \{x^{**} \in B^{**}\colon \langle x^*, x^{**}\rangle = \| x^* \|\}$ and that $\partial_{\|\cdot\|}(x^*) \cap E = \partial_{\|\cdot\|}(x^*) \cap B \neq \emptyset$ iff x^* attains its maximum on B. Therefore, $BP = \{x^* \in E^*\colon \partial_{\|\cdot\|}(x^*) \cap E \neq \emptyset\}$. It remains to apply the previous theorem. ■

Remark 6.5 As shown in [KG], Theorem 4.5, the last two statements are valid for all Banach spaces which admit an equivalent locally uniformly rotund norm.

7 Applications to best approximation theory

Let again E be a real Banach space with norm $\| \cdot \|$. Denote by S the unit sphere of E, i.e. the set $\{x \in E\colon \ \| x \| = 1\}$. Recall that the norm $\| \cdot \|$ in E is *locally uniformly rotund* if for every $x_0, x_n \in S$ such that $(1/2) \| x_0 + x_n \| \to 1$, it follows that $x_n \to x_0$. $\| \cdot \|$ is *strictly convex* if S does not contain line segments.

Let $A \subset E$ be a non-empty subset of E. The *metric projection* generated by A is the multivalued mapping $P_A\colon E \to A$ defined by $P_A(x)\colon = \{y \in A\colon \ \| x - y \| = \inf\{\| x - y' \|\colon y' \in A\}\}$. In 1966 Stečkin [St] stated the conjecture that the set $\{x \in E\colon P_A(x) = \emptyset$ or $P_A(x)$ is a singleton $\}$ is residual in E provided the norm in

E is strongly convex. Up to now there have been partial positive answers to this question (see [FaZh], [Kon1], [Kon2], [La], [St], [Za], [Zh1]).

Analogously, if A is a non-empty and bounded subset of E one can consider the *metric antiprojection* mapping $Q_A\colon E \to A$ defined by $Q_A(x)\colon = \{y \in A : \| x-y \| = \sup\{\| x - y' \| : y' \in A\}\}$. From the above point of view the metric antiprojection is investigated in [PaKa], [Zh1].

Using our selection theorems, we will prove here the original result of [St] about metric projections in a slightly stronger form containing Theorem 1.8 from [Zh1]. Before that we need the following lemma:

Lemma 7.1 *Let E be with locally uniformly rotund norm and A be its non-empty closed subset. Then the mapping P_A is a minimal closed graph and upper quasicontinuous mapping.*

Proof: The fact that $\mathrm{Gr}\,(P_A)$ is closed is verified easily having in mind that A is closed. In order to conclude, we will show that P_A satisfies condition c) from Proposition 3.1. Let $x_0 \in \mathrm{Dom}\,(P_A)$. We will show that for every $y \in P_A(x_0)$ the metric projection P_A is upper semicontinuous and single-valued at each point x from the interior of the line segment $[x_0, y]$. Since for every such x $P_A(x) = y$ we will easily get that P_A satisfies condition c) from Proposition 3.1.

Fix some $x \in [x_0, y]$. To prove that P_A is usc at x we will show the following (stronger) property: if $x_n \to x$ and $y_n \in P(x_n)$, then $y_n \to y$.

First, we may think, without loss of generality that the left-open segment $(x_0, y]$ is non-empty. If not, then $x_0 = y \in A$ and $P_A(x_0) = y$. Take some sequence $\{x_n\}$ converging to $x_0 = y$ and let $y_n \in P_A(x_n)$. Then, because the distance function is continuous, we get

$$\|y_n - y\| = \|y_n - x_0\| \leq \|y_n - x_n\| + \|x_n - x_0\| = d(x_n, A) + \|x_n - x_0\| \to d(x_0, A) = 0.$$

Hence P_A is usc at x_0.

So fix an element $x \in (x_0, y]$ and take some $y' \in A$. Then obviously $\|y' - x_0\| \geq \|y - x_0\|$ since y is a best approximation of x_0. But $\|y - x_0\| = \|y - x\| + \|x - x_0\|$, while $\|y' - x_0\| \leq \|y' - x\| + \|x - x_0\|$. Hence, $\|y' - x\| \geq \|y - x\|$ showing that $y \in P_A(x)$. Observe that the above arguments show that if y' is outside the closed ball $B[x_0; \|y - x_0\|]$ then it cannot be a best approximation of x since the inequality we got would be strict. Hence, if there are other best approximations of x in A they must be on the surface of the ball $B[x_0, \|y - x_0\|]$. But if y' is such that $y' \neq y$, $\|y' - x_0\| = \|y - x_0\|$ and $y' \in P_A(x)$ then we have:

$$\|y' - x_0\| = \|y - y_0\| = \|y - x\| + \|x - x_0\| = \|y' - x\| + \|x - x_0\|$$

which is a contradiction with the strict convexity of the norm.

Now, let $x_n \to x$ and $y_n \in P_A(x_n)$. Again by the continuity of the distance function we get:

$$\|y - x\| = d(x, A) = \lim d(x_n, A) = \lim \|y_n - x_n\| = \lim \|y_n - x\|$$

the last equality being true since $x_n \to x$. Observe that the points y_n are outside the ball $B(x_0, \|y - x_0\|)$ and are "tending" to the surface of the inner ball $B[x, \|y - x\|]$ which has only one common point with the bigger one–the point y. This together with local uniform rotundity of the space show that $y_n \to y$. Hence P_A is usc at x. ■

Theorem 7.2 *Let E have a locally uniformly rotund norm and A be closed (resp. closed and bounded). Then there exists a dense G_δ-subset E_1 of E at the points of which P_A (resp. Q_A) is usc and either empty or single-valued. Moreover, if* $\operatorname{Dom}(P_A)$ *(resp.* $\operatorname{Dom}(Q_A)$*) is dense in E then $E_1 \subset \operatorname{Dom}(P_A)$ (resp. $E_1 \subset \operatorname{Dom}(Q_A)$).*

Proof: We prove the theorem only for metric projections. The case of antiprojections is analogous.

Let $X\colon = \operatorname{Cl}\operatorname{Dom}(P_A)$. The set $E\backslash X$ is open and if it is non-empty then obviously for every $x \in E\backslash X$ $P_A(x) = \emptyset$ and P_A is usc at x. So consider $Y\colon = \operatorname{Int}\operatorname{Cl}\operatorname{Dom}(F)$. The set $X\backslash Y$ is nowhere dense in E. Hence, if $Y = \emptyset$, we are done. On the other hand, if $Y \neq \emptyset$ and one proves that there is a dense G_δ-subset X_1 of X such that P_A is single-valued and usc in X at each point of X_1, then one can easily get the conclusion of the theorem.

So, to finish, one consider P_A only in X and applies Theorem 5.16. ■

8 The Banach-Mazur game and optimization problems

Let X be a completely regular topological space and let us consider the family $C(X)$ of all continuous and bounded real-valued functions in X equipped with the usual sup-norm $\|f\| := \sup\{|f(x)| : x \in X\}$, $f \in C(X)$. Each $f \in C(X)$ determines in a natural way a minimization problem:

find $x_0 \in X$ with $f(x_0) = \inf\{f(x) : x \in X\} =: \inf(X, f)$.

We shall denote this problem by (X, f). Among the different properties of the minimization problem (X, f) the following ones are of special interest in the theory of optimization:

(a) (X, f) has a solution (existence of the solution);

(b) the solution set for (X, f) is a singleton (uniqueness of the solution) or is a compact set of X;

(c) the solution set has some sort of continuous dependence on the data of the problem (see bellow the precise definitions–this is called often stability of the solution set).

In general, taken together, the properties (a)-(c) for the problem (X, f) give the content of the notion "well-posed minimization problem (X, f)". More precisely, the minimization problem (X, f) is said to be *Tykhonov well-posed* if it has unique solution x_0 and $x_n \to x_0$ whenever $f(x_n) \to \inf(X, f)$. The sequences $\{x_n\} \subset X$ such that $f(x_n) \to \inf(X, f)$ are called *minimizing* for (X, f). For a continuous function f, the Tykhonov well-posedness of (X, f) simply means that every minimizing sequence is convergent. Since we are in a general topological space a question about minimizing nets may arise. It is easily seen that if the problem (X, f), $f \in C(X)$, is Tykhonov well-posed then every minimizing net (not only every minimizing sequence) converges to the unique solution of the problem. Therefore, in the case of Tykhonov well-posedness we can confine ourselves to the minimizing sequences and not to consider minimizing nets.

This is not the case with a generalized notion of this type when uniqueness of the solution is not required. The minimization problem (X, f), $f \in C(X)$, is said to be *generalized well-posed* if every minimizing net of the problem (X, f) has a convergent subnet. The generalized well-posedness of (X, f), $f \in C(X)$, implies that the solution set of the problem (X, f) (i.e. the set of minimizers of f) is nonempty and compact in X.

The well-posed problems are of special interest in Optimization both from theoretical and practical point of view. Indeed, it is seen that if the problem is (Tykhonov) well-posed then every numerical method for solving this problem which produces minimizing sequences will be successful in a sense that the produced iterative sequence will approach the (unique) minimum. On the other point of view, the well-posedness often means (see below Propositions 9.6) that the (unique) solution is stable under small perturbations of the data of the problem. I.e., we have continuous dependence of the solution on the data–this is important every time when we have to replace the original problem by a similar one which is close to it. For a comprehensive study of the these and other similar concepts of well-posedness we refer the reader to the book [DoZ].

The notion of well-posedness of a given problem (X, f), $f \in C(X)$, is also related to the differentiability properties of the sup-norm in $C(X)$ (see e.g. [ČKR1, ČKR2]. Let us mention that the concept of Tykhonov well-posedness for a problem (X, f) is termed also as f has a *strong minimum* (see e.g. [DvGZi1, DvGZi2]). Motivated by the case of linear functionals, one says in such a case also that f *strongly exposes* its unique minimum.

In general, a particular problem (X, f) may fail to have any of the properties (a)-(c). But the question which may be raised is to measure the "topological bigness" of the set of functions from $C(X)$ possessing some (or all) of the properties (a)-(c). The term "topological bigness" is understood (as above) in the Baire category sense. For instance, one may ask when the set $T := \{f \in C(X) : (X, f)$ is Tykhonov well-posed$\}$ is residual in $C(X)$. I.e., we are interested in under what conditions (necessary and sufficient) this set contains a dense G_δ-subset of $C(X)$. In other words, under what conditions almost all problems are well-posed? Similar questions for different classes of optimization problems have been a common point of interest for many authors in the last 30 years. We will see that the positive answer to the above question is closely related to the existence of residually defined selections of certain set-valued mappings. Moreover, investigating the above issue of generic well-posedness we will see that some of the sufficient conditions for the existence of the selections in Section 4 are also necessary. All this, on the other hand is intimately related to the existence of special winning strategies in the Banach-Mazur game $BM(X)$.

Analogously, one can consider the situations when the property from (a) or (b) is generic. In other words, situations when the set $E := \{f \in C(X) : (X, f)$ has a solution$\}$ or the set $U := \{f \in C(X) : (X, f)$ has unique solution$\}$ is residual in $C(X)$. We will see for example that the generic existence of solutions to the problems from $C(X)$ is a characterization of the existence of α-winning strategy in the Banach-Mazur game $BM(X)$.

All the above generic properties are a partial case of the following more general scheme of variational principle: let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued lower semicontinuous function, which is *proper*. The latter means that the *effective domain* $\operatorname{dom} f$ of f, i.e. the set $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$ is not empty. Equivalently, f is finite at least in one point. Let us recall that f is lower semicontinuous if its *epigraph*

$$\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} : f(x) \leq t\}$$

is closed in the product topology. In conclusion, f is lower semicontinuous and proper iff its epigraph is non-empty and closed in the product topology.

Let f be in addition bounded from below: then the notion of Tykhonov well-posedness for the minimization problem (X, f) is defined exactly as above–f has a unique minimum towards converges any minimizing sequence.

Suppose further that we are given a family of (usually at least continuous) bounded from below functions $\mathcal{P}$ defined in X. We may think that the class $\mathcal{P}$ is equipped with some complete metric (most often a $\mathcal{P}$ is a Banach space lying in $C(X)$ with a norm which is at least as stronger as the sup-norm in $C(X)$). The following question is of importance in many situations: does there exist a non-empty subset $\mathcal{P}'$ of $\mathcal{P}$ so that for any $g \in \mathcal{P}'$ the minimization problem $(X, f+g)$ attains its minimum in X (or stronger, is Tykhonov well-posed). In other words we are seeking whether in the class $\mathcal{P}$ there are perturbations g making the perturbed minimization problem $(h + f)$ solvable (or well-posed). Often, the aim is to prove that the set $\mathcal{P}$ is as big as possible in $\mathcal{P}$, i.e. is dense in $\mathcal{P}$ or even more, contains a dense and G_δ-subset of $\mathcal{P}$. These are the general schemes of Ekelenad variational principle [Ek1, Ek2], Deville-Godefroy-Zizler [DvGZi1, DvGZi1] variational principle, Stegall variational principle [S1] and Borwein-Preiss [BoPr] smooth variational principles. Our generic existence (or well-posedness) described above is evidently a partial case of this scheme with $f \equiv 0$ and the class $\mathcal{P}$ being just $C(X)$ with the usual norm.

Such variational principles have turned out to be extremely useful in different branches of mathematics, as optimization, non-linear analysis, critical point theory, differentiability of convex functions, existence of solutions to Hamilton-Jacobi equations and many others.

9 The solution mapping in $C(X)$

We introduce below a mapping which will play an important role in the sequel and prove its basic properties.

Let X be a completely regular topological space and $C(X)$ be the space of all continuous and bounded real-valued functions in X equipped with the sup-norm. In $C(X)$ we define the *solution mapping* $M : C(X) \to X$ related to the minimization problems from $C(X)$ by

$$M(f) := \{x \in X : f(x) = \inf(X, f)\}, \quad f \in C(X).$$

Obviously M is onto. Moreover, the following facts are true for M.

Proposition 9.1 *The mapping M has the following properties:*

(a) $\mathrm{Gr}\,(M)$ *is a closed subset of* $X \times Y$*;*

(b) $\mathrm{Dom}\,(M)$ *is dense in* $C(X)$*;*

(c) M is open;

(d) for every open U in X the set $IntM^{\#}(U)$ is dense in $M^{\#}(U) \bigcap \mathrm{Dom}\,(M)$*;*

(e) for every two open sets W in $C(X)$ and U in X with $W \bigcap M^{-1}(U) \neq \emptyset$ there is a non-empty open $W' \subset W$ such that $M(W') \subset U$.

Proof: (a) is trivial. As to (b), let $f \in C(X)$ and $\varepsilon > 0$ be arbitrary. Then, obviously $M(f_\varepsilon) \neq \emptyset$ for $f_\varepsilon(x) := \sup\{f(x), \inf(X, f) + \varepsilon\}$.

Further we demonstrate (c). Let W be an open subset of $C(X)$ and $x_0 \in M(f_0)$ for some $f_0 \in W$. Take $\varepsilon > 0$ such that the ball $B(f_0, \varepsilon) \colon = \{f \in C(X) \colon \|f - f_0\| < \varepsilon\} \subset W$. Then each $x' \in \{x \in X \colon f_0(x) < \inf(X, f_0) + \varepsilon\}$ is a minimizer of some f from W, e.g. of the function f_{0_ε} considered above.

It is easy to see that (d) is a consequence of (e). So we prove (e).

Let $x_0 \in M(f_0) \bigcap U$ for some $f_0 \in W$. Since X is completely regular there exists a function $h_0 \in C(X)$ such that $h_0(x_0) = 0$, $h_0(X \setminus U) = 1$ and $\|h_0\| = 1$. Find $\delta > 0$ such that $f_0 + \delta h_0 \in W$. Let further, $W' \subset W$ be an open set in $C(X)$ containing $f_0 + \delta h_0$ and such that $\mathrm{diam}(W') < \delta/3$. Take $f \in W'$. Since for $x \in X \setminus U$ one has

$$\begin{aligned} f(x) &\geq (f_0 + \delta h_0)(x) - \frac{\delta}{3} = f_0(x) + \frac{2\delta}{3} \\ &\geq f_0(x_0) + \frac{2\delta}{3} = (f_0 + \delta h_0)(x_0) + \frac{2\delta}{3} \\ &> (f_0 + \delta h_0)(x_0) + \frac{\delta}{3} \geq f(x_0), \end{aligned}$$

we see that $M(f) \subset U$.

The proof of the proposition is completed. ■

Immediate corollaries of this proposition and Proposition 3.1 and Proposition 3.2 are the following two ones:

Corollary 9.2 *Let X be completely regular. Then the solution mapping $M : C(X) \to X$ is an open minimal closed graph upper quasicontinuous mapping.*

Corollary 9.3 *Let X be compact. Then the solution mapping $M : C(X) \to X$ is an open minimal usco.*

Further we show that the continuity-like properties of M are related to the concept of well-posedness. First we investigate the relationship between the generalized well-posedness and upper semicontinuity of M. Namely, we have

Proposition 9.4 *Let X be a completely regular topological space. The minimization problem (X, f), $f \in C(X)$, is generalized well-posed iff the solution mapping $M : C(X) \to X$ is usco at f.*

Proof: Let (X, f) be generalized well-posed. Then, as it was mentioned, $M(f)$ is non-empty and compact. Suppose M is not usc at f. Then there is an open set V of X with $M(f) \subset V$ such that for every open U of $C(X)$ containing f one has $M(U) \setminus V \neq \emptyset$. Let $x_U \in M(U) \setminus V$. Then $x_U \in M(f_U)$ for some $f_U \in U$. Ordering the family of all open neighborhoods U of f by inclusion we get two nets $\{f_U\} \subset C(X)$ and $\{x_U\} \subset X$ such that $f_U \to f$ and $x_U \in M(f_U)$.

Further, it is easily checked that the *marginal function* $\inf(X, \cdot)$ is a continuous function from $C(X)$ into the reals $\mathbb{R}$. Hence $f_U(x_U) \to \inf(X, f)$. This, together with $f_U \to f$ gives $f(x_U) \to \inf(X, f)$. Hence, $\{x_U\}$ is a minimizing net for the minimization problem (X, f). But this net does not have any subnet convergent to a solution of (X, f) since $x_U \notin V$ and $M(f) \subset V$. This is a contradiction.

Conversely, let M be usco at $f \in C(X)$. Take a minimizing net $\{x_\lambda\}$ for the problem (X, f) and consider the continuous and bounded real-valued functions $f_\lambda(x) := \sup\{f(x), f(x_\lambda)\}$. Since $f(x_\lambda) \to \inf(X, f)$ we get $f_\lambda \to f$ in $C(X)$. Hence, by the upper semicontinuity of M at f for every open V in X containing $M(f)$ we have $M(f_\lambda) \subset V$ for large λ, giving $x_\lambda \in V$ for large λ. Now, it is a routine matter to organize a subnet of $\{x_\lambda\}$ converging to a point from the compact set $M(f)$. The proof is completed. ■

Another fact characterizing the generalized well-posedness is related with the compactification of X. Let βX as usual denote the Stone-Čech compactification of the completely regular topological space X. Given $f \in C(X)$ we will denote by $e(f)$ the unique continuous extension of f in βX. Let $\tilde{M}$ be the solution mapping determined by βX, i.e. the set-valued mapping between $C(\beta X)$ and βX which assigns to each function from $C(\beta X)$ its minimizers in βX. Since, in fact, $C(\beta X)$ and $C(X)$ are congruent (which means that as Banach spaces they are the same) here (and in the sequel) we will always consider the mapping $\tilde{M}$ to act between $C(X)$ and βX. I.e. for $f \in C(X)$, $\tilde{M}(f) = \{x \in \beta X : e(f)(x) = \inf(X, f) = \inf(\beta X, e(f))\}$. According to Corollary 9.3 the mapping $\tilde{M}$ is an open minimal usco. Now we have:

Proposition 9.5 *Let X be a completely regular topological space and $f \in C(X)$. Then the minimization problem (X, f) is generalized well-posed iff $\tilde{M}(f) \subset X$.*

Proof: Let (X, f), $f \in C(X)$, be generalized well-posed and suppose that there is a point $x \in \beta X \setminus X$ so that $e(f)(x) = \inf(\beta X, f) = \inf(X, f)$. Then there is a net $\{x_\lambda\} \subset X$ which converges to x. But then $f(x_\lambda) = e(f)(x_\lambda) \to e(f)(x)$ and hence $\{x_\lambda\}$ is minimizing net for the problem (X, f) which does not have a convergent (in X) subnet. This is a contradiction.

Conversely suppose that $\tilde{M}(f) \subset X$. Let M be the solution mapping determined by X. Since obviously $M(f) = \tilde{M}(f) \cap X$, and $\tilde{M}$ is usco, we see easily that M is usco at f. According to Proposition 9.4 this means that (X, f) is generalized well-posed. ■

Since it is clear that a problem $(X, f), f \in C(X)$, is Tykhonov well-posed iff (X, f) is generalized well-posed and has unique solution, then the following two assertions are immediate consequence from the above propositions.

Proposition 9.6 *Let X be a completely regular topological space and $f \in C(X)$. The minimization problem (X, f), $f \in C(X)$, is Tykhonov well-posed iff the solution mapping $M : C(X) \to X$ is single-valued and usc at f.*

Proposition 9.7 *Let X be a completely regular topological space and $f \in C(X)$. Then the minimization problem (X, f) is Tykhonov well-posed iff $\tilde{M}(f)$ is a singleton lying in X.*

10 Winning strategies in the Banach-Mazur game and generic existence of solutions

In this section we give a characterization of the existence of α-winning strategy in the Banach-Mazur game $BM(X)$, where X is a completely regular topological space. Before stating it we need a simple fact concerning multivalued mappings.

Proposition 10.1 *Let $F : X \to Y$ be a multivalued mapping with closed graph. Let $x_0 \in X$ and the family $\{U_\lambda : \lambda \in A\}$ be a local base for x_0 in X. Then*

$$F(x_0) = F(\bigcap\{U_\lambda : \lambda \in A\}) = \bigcap\{F(U_\lambda) : \lambda \in A\} = \bigcap\{\operatorname{Cl} F(U_\lambda) : \lambda \in A\}$$

.

Proof: The next chain of equalities and inclusions is clear:

$$F(x_0) = F(\bigcap\{U_\lambda : \lambda \in A\}) \subset \bigcap\{F(U_\lambda) : \lambda \in A\} \subset \bigcap\{\operatorname{Cl} F(U_\lambda) : \lambda \in A\}.$$

So, take some $y \in \bigcap\{\operatorname{Cl} F(U_\lambda) : \lambda \in A\}$ and suppose $y \notin F(x_0)$. Hence, $(x_0, y) \notin \operatorname{Gr}(F)$ and since F has a closed graph, there are open sets $U \subset X$ and $V \subset Y$ such that $x_0 \in U$, $y \in V$ and $F(U) \bigcap V = \emptyset$. But the family $\{U_\lambda : \lambda \in A\}$ is a base for x_0 in X. Therefore, there exists $\lambda_0 \in A$ with $U_{\lambda_0} \subset U$. On the other hand, $y \in \operatorname{Cl} F(U_{\lambda_0})$ which entails $V \bigcap F(U) \neq \emptyset$. This is a contradiction which completes the proof. ■

Now we are ready to prove the following characterization of weakly α-favorable spaces:

Theorem 10.2 *For a completely regular topological space X the following assertions are equivalent:*

(a) X is weakly α-favorable;

(b) every (demi-) open and minimal closed graph upper quasicontinuous mapping $F : Z \to X$ with dense domain acting from a complete metric space Z into X is non-empty valued at the points of a dense and G_δ-subset of Z;

(c) the set $E = \{f \in C(X) : (X, f)$ has a solution$\}$ contains a dense G_δ-subset of $C(X)$.

Proof: Suppose (a) is true and let $F : Z \to X$ be a demi-open minimal closed graph upper quasicontinuous mapping from the complete metric space Z into X so that $\operatorname{Dom}(F)$ is dense in Z. We may suppose that $F(Z)$ is dense in X, otherwise we may consider $F : Z \to \operatorname{Cl} F(Z)$ which is again a demi-open minimal closed graph upper quasicontinuous mapping. Observe that, because F is demi-open, for every non-empty open W of Z we have $\operatorname{Int} \operatorname{Cl} F(W) \neq \emptyset$.

Let s be a winning strategy for the player α in the game $BM(X)$. The next lemma is modification of Lemma 4.8.

Lemma 10.3 *Let F be as in Theorem 10.2, $(U_1, V_1, \dots, U_n, V_n)$, $n \geq 1$, be a partial play in the game $BM(X)$ and W_n is a non-empty open subset of Z such that* $\mathrm{Int}\,\mathrm{Cl}\,F(W_n) \subset V_n$. *Let $\Gamma(W_n)$ be a maximal family of triples $(U_{n+1}, V_{n+1}, W_{n+1})$ such that:*

(a) U_{n+1} is a non-empty open subset of $\mathrm{Int}\,\mathrm{Cl}\,F(W_n)$*;*

(b) $V_{n+1} = s(U_1, V_1, \dots, U_n, V_n, U_{n+1})$;

(c) W_{n+1} is a non-empty subset of Z such that $\mathrm{diam}\,(W_{n+1}) < 1/(n+1)$, $\mathrm{Cl}\,(W_{n+1}) \subset W_n$ *and* $\mathrm{Int}\,\mathrm{Cl}\,F(W_{n+1}) \subset V_{n+1}$*;*

(d) the family $\gamma(W_n) \colon = \{W_{n+1} \colon (U_{n+1}, V_{n+1}, W_{n+1}) \in \Gamma(W_n)$ for some U_{n+1}, $V_{n+1}\}$ is disjoint;

Then the set $H(W_n) \colon = \bigcup\{W_{n+1} \colon W_{n+1} \in \gamma(W_n)\}$ is dense in W_n.

Proof of the Lemma: A family satisfying (a)-(d) always exists. Take a maximal one $\Gamma(W_n)$ and suppose the conclusion of the lemma is not true. Then there exists a non-empty open subset G of Z with $G \subset W_n$ and $G \bigcap H(W_n) = \emptyset$. Since F is demi-open $\mathrm{Int}\,\mathrm{Cl}\,F(G) \neq \emptyset$. Moreover, $\mathrm{Int}\,\mathrm{Cl}\,F(G) \subset \mathrm{Int}\,\mathrm{Cl}\,F(W_n) \subset V_n$. Let $U_{n+1} \colon = \mathrm{Int}\,\mathrm{Cl}\,F(G)$ and $V_{n+1} := s(U_1, V_1, \dots, U_n, V_n, U_{n+1})$. By Proposition 3.1 (c) and the regularity of X there is a non-empty open subset W_{n+1} of Z such that $W_{n+1} \subset G$ and $\mathrm{Int}\,\mathrm{Cl}\,F(W_{n+1}) \subset V_{n+1}$. We may think, in addition, that $\mathrm{Cl}\,(W_{n+1}) \subset W_n$ and $\mathrm{diam}\,(W_{n+1}) < 1/(n+1)$. Now, the family $\Gamma(W_n) \bigcup \{(U_{n+1}, V_{n+1}, W_{n+1})\}$ is strictly larger than $\Gamma(W_n)$ and satisfies (a)-(d). This is a contradiction showing that the maximal $H(W_n)$ is dense in (W_n). ■

Let us mention that, as usual, the above lemma is true also for $n = 0$ provided we put $U_0 = V_0 = X$.

Now, we get back to the proof of the theorem. Proceeding as in the proof of Theorem 4.3 we obtain a sequence of families $\{\Gamma_n\}_{n \geq 1}$ of triples and a sequence of disjoint families $\{\gamma_n\}_{n \geq 0}$ of open sets in Z, with $\gamma_0 = \{Z\}$, such that for every $n \geq 1$ we have:

(i) Γ_n is a union of the families $\Gamma(W_{n-1})$, $W_{n-1} \in \gamma_{n-1}$, where $\Gamma(W_{n-1})$ is obtained by the Lemma from some uniquely determined partial play $(U_1, V_1, \dots, U_{n-1}, V_{n-1})$;

(ii) γ_n is a union of the families $\gamma(W_{n-1})$ from condition (d) of the Lemma;

(iii) the set $H_n \colon = \bigcup\{W_n \colon W_n \in \gamma_n\}$ is open and dense in Z.

Let $H_0 \colon = \bigcap_{n=1}^{\infty} H_n$. Obviously H_0 is a dense G_δ-subset of Z. Take $z_0 \in H_0$. By the properties above, this z_0 determines a unique sequence $\{W_n\}_{n \geq 1}$ such that for every $n \geq 1$, $W_n \in \gamma_n$, $z_0 \in W_n$, $\mathrm{Cl}\,(W_{n+1}) \subset W_n$ and $\mathrm{diam}\,(W_n) < 1/n$. Hence $\{z_0\} = \bigcap_{n=1}^{\infty} W_n$ and the family $\{W_n\}_{n=1}^{\infty}$ is a local base for z_0 in Z. By the properties (a)-(d) from Lemma 10.3 and conditions (i)-(iii) above it follows that there is an s-play $p = \{U_n, V_n\}_{n=1}^{\infty}$ such that $U_{n+1} \subset \operatorname{Int}\operatorname{Cl} F(W_n) \subset V_n$ for every $n \geq 1$.

Hence, by Proposition 10.1 we have

$$F(z_0) = \bigcap_{n=1}^{\infty} \operatorname{Int}\operatorname{Cl} F(W_n) = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n = T(p).$$

Since s is an α- winning strategy, we see that $F(z_0) = T(p) = \bigcap_{n=1}^{\infty} V_n \neq \emptyset$. The proof of the implication (a)$\Rightarrow$(b) is completed.

The implication (b)$\Rightarrow$(c) follows by the fact the solution mapping $M : C(X) \to X$ is open and minimal closed graph upper quasicontinuous mapping. So we prove (c)$\Rightarrow$(a).

Let (c) be fulfilled and consider the solution mapping $M : C(X) \to X$. The set $\operatorname{Dom}(M)$ contains a dense G_δ-subset of $C(X)$. Then, there exist countably many open and dense subsets $\{G_n\}_{n\geq 1}$ of $C(X)$ such that $\bigcap_{n=1}^{\infty} G_n \subset \operatorname{Dom}(M)$. The sets $F_n \colon = C(X) \setminus G_n, n \geq 1$, are closed and nowhere dense in $C(X)$. That is $\operatorname{Int}(F_n) = \emptyset$ for every $n \geq 1$.

We start now constructing a winning strategy for the player α in $BM(X)$. Let U_1 be a non-empty open subset of X. Consider the set $\operatorname{Int} M^{\#}(U_1)$ which is non-empty by Proposition 9.1 (d). Since F_1 is closed and nowhere dense in $C(X)$, the set $\operatorname{Int} M^{\#}(U_1) \setminus F_1$ is non-empty and open in $C(X)$. Take an open ball B_1 in $C(X)$ with radius less or equal to 1, such that $B_1 \subset \operatorname{Int} M^{\#}(U_1) \setminus F_1$. Define now the value of the strategy s at U_1 by $s(U_1) \colon = M(B_1)$. By Proposition 9.1 (c), $s(U_1)$ is a non-empty open subset of U_1.

Further, let U_2 be an arbitrary non-empty open subset of $V_1 = s(U_1) = M(B_1)$. Since $U_2 \subset M(B_1)$ there is some $f \in B_1$ such that $M(f) \bigcap U_2 \neq \emptyset$. Hence, by Proposition 9.1 (e) there exists a non-empty open $W \subset B_1$ such that $M(W) \subset U_2$. As above the set $W \setminus F_2$ is a non-empty and open subset of $C(X)$. Take an open ball B_2 with radius less or equal to $1/2$ so that $\operatorname{Cl}(B_2) \subset W \setminus F_2 \subset B_1$ and put $s(U_1, V_1, U_2) \colon = M(B_2)$. Obviously $s(U_1, V_1, U_2)$ is a non-empty open subset of U_2. Proceeding by induction we define the strategy s for every chain $(U_1, V_1, \dots, U_n)$, $n \geq 1$, such that $U_k \subset V_{k-1}$ and $V_{k-1} = s(U_1, V_1, \dots, U_{k-1})$ for every k, $1 \leq k \leq n$ (with $V_0 = X$).

Let $p = \{U_n, V_n\}_{n=1}^{\infty}$ be an s-play and $\{B_n\}_{n\geq 1}$ is the sequence of open balls in $C(X)$ associated with $\{U_n\}_{n\geq 1}$ and $\{V_n\}_{n\geq 1}$ from the construction of s. Then for every $n \geq 1$:

1) $\operatorname{Cl}(B_{n+1}) \subset B_n$ and $B_n \bigcap F_n = \emptyset$;

2) $\operatorname{diam}(B_n) < 1/n$;

3) $V_n = M(B_n)$

The conditions 1) and 2) above guarantee that $\bigcap_{n=1}^{\infty} B_n$ is a one-point set in $C(X)$, say f_0, and $\{B_n\}_{n=1}^{\infty}$ is a local base for f_0. Moreover, 1) shows in addition that $f_0 \in C(X) \setminus \bigcup_{n=1}^{\infty} F_n \subset \mathrm{Dom}\,(M)$. Therefore, by 3) and Proposition 10.1 we have

$$\emptyset \neq M(f_0) = \bigcap_{n=1}^{\infty} M(B_n) = \bigcap_{n=1}^{\infty} V_n = T(p).$$

Hence s is a winning strategy for the player α in the Banach–Mazur game $BM(X)$. This completes the proof. ■

11 Strengthened strategies in the Banach-Mazur game and generic well-posedness

In this section we consider stronger conditions for the player α to win in the game $BM(X)$. As a result we obtain special kind of winning strategies for the player α which already were considered in the previous chapter. On the one hand, it turns out that the existence of such kind of strategies characterizes the generic well-posedness in the class $C(X)$. On the other hand we will see that, those strengthened strategies are not only sufficient for the existence of residually defined selection as it was shown in Section 4 but are also necessary.

Indeed, the following two theorems characterize the generic well-posedness in the class $C(X)$. Simultaneously they characterize also the existence of residually defined selections for (demi-)open lower demicointinuous and minimal pseudo usc mappings.

Theorem 11.1 *Let X be a completely regular topological space. The following assertions are equivalent:*

(a) the space X is almost Čech complete;

(b) the player α has a complete (stationary) winning strategy in the game $BM(X)$;

(c) every demi-open lower demicontinuous mapping $F : Z \to X$ with closed graph (and dense domain), acting from a Baire space Z into X possesses an usco selection $G : Z_1 \to X$ where Z_1 is a dense G_δ-subset of Z with $Z_1 \subset \mathrm{Dom}\,(F)$;

(d) every (demi-)open minimal closed graph upper quasicontinuous mapping $F : Z \to X$, acting from a Baire space Z into X is usco (hence, also non-empty valued) at the points of a dense G_δ-subset Z_1 of Z;

(e) the set $GT\colon = \{f \in C(X)\colon (X, f)$ is generalized well-posed $\}$ contains a dense G_δ-subset of $C(X)$.

Theorem 11.2 *Let X be a completely regular topological space. The following assertions are equivalent:*

(a) the space X contains a dense completely metrizable subspace;

(b) the player α has a complete (stationary) winning strategy s in the game $BM(X)$ such that for every s-play $p = \{U_n, V_n\}_{n=1}^{\infty}$ the target set $T(p)$ is a singleton;

(c) every demi-open lower demicontinuous mapping $F : Z \to X$ with closed graph (and dense domain), acting from a Baire space Z into X possesses a single-valued and continuous selection $g : Z_1 \to X$ where Z_1 is a dense G_δ-subset of Z with $Z_1 \subset \mathrm{Dom}\,(F)$;

(d) every (demi-)open minimal closed graph upper quasicontinuous mapping $F :$ $Z \to X$, acting from a Baire space Z into X is single-valued and usc at the points of a dense G_δ-subset Z_1 of Z;

(e) the set $T\colon = \{f \in C(X)\colon (X, f)$ is Tykhonov well-posed $\}$ contains a dense and G_δ-subset of $C(X)$.

We will first give the proof of the second theorem. The proof of the previous one is similar.

Proof of Theorem 11.2: (a)⇒(b) is Proposition 5.3, while (b)⇒(c) is Theorem 5.4. (c)⇒(d) follows by Thoerem 5.15 Finally, (d)⇒(e) is a consequence of Propostion 9.1 and Proposition 9.6. So it remains to prove (e)⇒(a).

Suppose that T contains a dense G_δ-subset T_1 of $C(X)$. Consider the Čech-Stone compactification βX of X and let $\tilde{M}$ be the corresponding solution mapping between $C(X)$ and βX, i.e. given $f \in C(X)$, $\tilde{M}(f)$ are the minimizers of the (unique) continuous extension $e(f)$ of f in βX. Consider further the mapping $\tilde{M}^{-1}\colon \beta X \to C(X)$. It is lsc (since by Proposition 9.1 M is open) and has closed graph, again by the same proposition. Moreover, since $\tilde{M}$ is minimal, then it is lower demicontinuous and thus (Proposition 5.1 $\tilde{M}^{-1}$ is demi-open. Hence, by Theorem 5.4 there exist a dense G_δ-subset X_1 of βX and a continuous single-valued mapping $h\colon X_1 \to T_1$ which is a selection of $\tilde{M}^{-1}$ on X_1. Obviously X_1 is Čech complete. Take $x \in X_1$. Then $h(x) \in T_1$ and hence $h(x) \in T$. Therefore $\tilde{M}(h(x)) \subset X$, showing that $x \in X$. Hence $X_1 \subset X$. On the other hand, since h is a selection of M^{-1} mapping X_1 into T_1 and M is usco and single-valued on T, it is seen that h is a homeomorphism between X_1 and $h(X_1) \subset T_1$. To finish, let us recall that every metrizable Čech complete space is completely metrizable. ■

Proof of Theorem 11.1:

(a)⇒(b) is Proposition 5.3, (b)⇒(c) is Theorem 5.4, (c)⇒(d) follows by Thoerem 5.15. As above, (d)⇒(e) is a consequence of Proposition 9.1 and Proposition 9.4. To prove (e)⇒(a) one proceeds as above (the proof of the same implication) and obtain a dense subset X_1 of X which turns out to be Čech complete.

Remark 11.3 Since for any $x \in \beta X, \tilde{M}^{-1}(x)$ is a convex closed subset of $C(X)$ and $\tilde{M}^{-1}$ is lsc in βX, the classical Michael selection theorem [M1] always gives a single-valued selection of $\tilde{M}^{-1}\colon \beta X \to C(X)$ which is defined on βX. The values of this selection, however, are not obliged to lie in T or GT.

Final Remark: The material in these notes is based on the content of the following papers from the references below: [ČKR1, ČKR2, ČKR3, ČKR4, ČKR5, ČKR6, KMoR1, KMoR2, KR1, KR2, KR3, KR4, R1, R2, R3].

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